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Diffraction by *m*-bonacci gratings

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Abstract

We present a simple diffraction experiment with *m*-bonacci gratings as a new interesting generalization of the Fibonacci ones. Diffraction by these non-conventional structures is proposed as a motivational strategy to introduce students to basic research activities. The Fraunhofer diffraction patterns are obtained with the standard equipment present in most undergraduate physics labs and are compared with those obtained with regular periodic gratings. We show that *m*-bonacci gratings produce discrete Fraunhofer patterns characterized by a set of diffraction peaks which positions are related to the concept of a generalized golden mean. A very good agreement is obtained between experimental and numerical results and the students' feedback is discussed.

Keywords: diffraction, Fibonacci, aperiodic sequence

(Some figures may appear in colour only in the online journal)

1. Introduction

Diffraction is a standard topic in teaching wave optics in physics courses at the university level [1]. This phenomenon appears when light interacts with an obstacle or aperture with a size comparable to its wavelength. It plays an important role, for instance, in image forming systems due to the finite sizes of lenses and other optical components. Typically, a single slit, double slits, or periodic gratings are classical examples of objects employed to teach this phenomena. Interesting educational software has been recently presented to give students the possibility of analysing many different configurations in an easy way [2]. On the other hand, some creative experimental works have been published to increase the interest of students. For example, the tracks of a compact disc [3], the spiral element of a spider orb-web [4], the cells of the onion epidermis [5], and the array of pixels in CCD image sensors or LCD screens [6] have been used as non-conventional diffraction gratings.

Other non-traditional diffractive objects such fractals are also very motivational because they offer a way to introduce students to basic research activities [7]. Between the complete order (periodic) and disorder (non-periodic), many other aperiodic distributions following deterministic sequences also display characteristic discrete Fourier spectra [8]. In particular, Fibonacci gratings are also one of the most common examples of aperiodic structures exhibiting two incommensurate periods (i.e., they are quasiperiodic). The ratio of these two periods is equal to an irrational number known as the golden mean. This number has been historically associated with the subjective concepts of equilibrium, harmony, or beauty and can be found profusely in nature [9]. In technology, some examples of structures based on the Fibonacci sequence are the nanostructured semiconductor superlattices [10] or the quasi-periodic metamaterial multilayers [11]. In physics study, some other interesting examples where the Fibonacci sequence and the golden mean appears include the study of a network of resistors [12] and capacitors [13] or the coupled-oscillator problem [14]. In the case of a one-dimensional Fibonacci grating, the Fourier spectrum produced by this optical element comprises a set of discrete diffraction peaks which relative positions are defined by the golden mean [15]. This distinctive property has been also obtained with other advanced diffractive optical components, for example with different kinds of diffractive lenses based on the Fibonacci sequence [16].

In line with our previous work, we consider in this paper the *m*-bonacci grating as an interesting generalization of the Fibonacci ones. In fact, Fibonacci is the particular case of a *m*-bonacci sequence with $m = 2$ [17]. We present here a simple experiment (from a didactic point of view) to verify the discrete Fourier spectra produced by these diffractive optical elements with the standard equipment present in most undergraduate physics laboratories. For comparison, the diffraction patterns produced by regular periodic gratings are also obtained. We show numerically and experimentally that the diffraction peaks generated by *m*-bonacci gratings are related to the concept of a generalized golden mean.

The outline of the paper is the following. In section 2, the new aperiodic gratings design is given. In section 3, the basic theory is presented, including analytical expressions for the diffraction patterns, and simulations for different structural parameters. Section 4 describes a classroom experiment with *m*-bonacci gratings employed in electronic engineering undergraduate courses at the Technical University of Valencia (Spain) to teach diffraction properties. Finally, in section 5, the students' feedback is given and some conclusions are drawn.

2. *m*-bonacci diffraction gratings design

A *m*-bonacci grating is defined as a set of slits distributed aperidically according to the following recursive procedure. Starting with m elements (seeds), $N_{m,0} = 0, N_{m,1} = 1$, and $N_{m,j} = \sum_{i=1}^{j} N_{m,j-i}$ with $1 < j < m$, the *m*-bonacci numbers $\{N_{m,j}\}$ are obtained by the iterative rule $N_{m,S} = \sum_{i=1}^{m} N_{m,S-i}$ with $S \geqslant m$. For example, the well-known Fibonacci numbers (*m*-bonacci numbers with $m = 2$) result from the sum of the two preceding numbers [18], so $N_{2,S} = N_{2,S-1} + N_{2,S-2}$ for $S \geqslant 2$ with $N_{2,0} = 0$ and $N_{2,1} = 1$, resulting $N_{2,i} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots\}$. Tribonacci numbers ($m = 3$) are like Fibonacci numbers, but the sequence starts with three predetermined terms $N_{3,0} = 0$, $N_{3,1} = 1$, and $N_{3,2} = 1$. Besides, each Tribonacci number is the sum of the three preceding three ones,

$N_{3,S} = N_{3,S-1} + N_{3,S-2} + N_{3,S-3}$ for $S \geqslant 3$, so $N_{3,i} = \{0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13, 24,\ 44,\ 81\ldots\}$. The generalized golden mean or golden ratio is defined as the limit of the ratio of two consecutive m-bonacci numbers

$$\varphi_m = \lim_{S\to\infty}\left(\frac{N_{m,S}}{N_{m,S-1}}\right). \tag{1}$$

Taking into account that the m-bonacci can be expressed as the sum of the m preceding numbers, it is easy to prove that the preceding limit results in the transcendental equation

$$(\varphi_m)^m - \sum_{i=1}^{m}(\varphi_m)^{m-i} = 0. \tag{2}$$

Solving this equation we obtain $\varphi_2 = \frac{1}{2}(1 + \sqrt{5}) = 1.618$ for $m = 2$ (Fibonacci) and $\varphi_3 = \frac{1}{3}(1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}) = 1.839$ for $m = 3$ (Tribonacci).

In a similar way, a binary m-bonacci sequence can also be generated with m seed elements, $t_{m,0} = \{0\},\ t_{m,1} = \{1\}$, and $t_{m,j} = \{t_{m,j-1}t_{m,j-2}\ldots t_{m,0}\}$ with $1 < j < m$, and the successive elements of the sequence are obtained as the concatenation of the m previous ones, $t_{m,S} = \{t_{m,S-1}t_{m,S-2}\ldots t_{m,S-m}\}$ with $S \geqslant m$. Particularly, the Fibonacci sequence ($m = 2$) is defined by the seed $t_{2,0} = \{0\},\ t_{2,1} = \{1\}$, and the concatenation rule $t_{2,S} = \{t_{2,S-1}t_{2,S-2}\}$ with $S \geqslant 2$, so:

$$\begin{aligned}
&t_{2,2} = \{10\},\\
&t_{2,3} = \{101\},\\
&t_{2,4} = \{10110\},\\
&t_{2,5} = \{10110101\},\\
&t_{2,6} = \{1011010110110\},\ \ldots
\end{aligned}$$

Note that $t_{2,6}$ presents $N_{2,6} = 8$ type-1 elements and $N_{2,5} = 5$ type-0 elements, being the total number of elements $N_{2,7} = 13 = 8 + 5$. In a similar way, for the construction of the Tribonacci sequence ($m = 3$) we used the seed elements $t_{3,0} = \{0\},\ t_{3,1} = \{1\}, t_{3,2} = \{10\}$, and the recursive relation $t_{3,S} = \{t_{3,S-1}t_{3,S-2}t_{3,S-3}\}$ with $S \geqslant 3$, so

$$\begin{aligned}
&t_{3,3} = \{1010\},\\
&t_{3,4} = \{1010101\},\\
&t_{3,5} = \{1010101101010\},\\
&t_{3,6} = \{101010110101010101011010\},\ \ldots
\end{aligned}$$

Note again that $t_{3,6}$ presents $N_{3,7} = 13$ type-1 elements and $N_{3,5} + N_{3,4} = 7 + 4 = 11$ type-0 elements, being the total number of elements $N_{3,7} = 24 = 13 + 7 + 4$. In general, the m-bonacci sequence at an arbitrary generation level S contains $N_{m,S}$ type-1 elements and $\sum_{i=1}^{m-1} N_{m,S-i} = N_{m,S+1} - N_{m,S}$ type-0 elements, so the total number of elements is $N_{m,S+1} = \sum_{i=0}^{m-1} N_{m,S-i}$. Then, the ratio ($\tau_m$) between the number of type-1 and type-0 elements is $\tau_m = \frac{N_{m,S}}{N_{m,S+1} - N_{m,S}} \approx \frac{1}{\varphi_m - 1}$ when $S \to \infty$. In the particular cases of the Fibonacci and Tribonacci sequences, $\tau_2 \approx \frac{1}{\varphi_2 - 1} = \varphi_2 \approx 1.618$ and $\tau_3 \approx \frac{1}{\varphi_3 - 1} \approx 1.191$, respectively.

With this encoding, an m-bonacci grating at a given generation level, S, can be constructed as a sequence of $N_{m,S+1}$ slits with the same width a, but defining the transmittance $t_{m,S,l}$ of the lth slit as the lth element of the binary array $t_{m,S}$, where $t_{m,S,l} = 1$ for transparent zones (slits) and $t_{m,S,l} = 0$ for opaque zones. Then, an m-bonacci grating presents $N_{m,S}$ transparent slits and $N_{m,S+1} - N_{m,S}$ opaque zones aperiodically distributed, as shown in

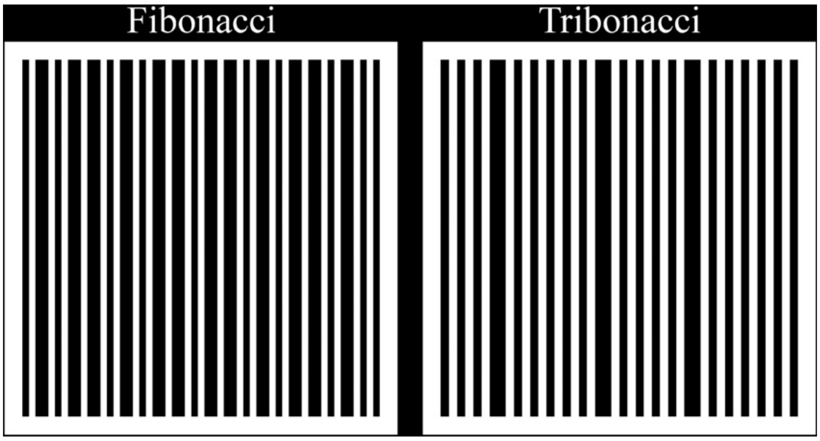

Figure 1. Aperiodic gratings based on the m-bonacci sequence with $m = 2$ (Fibonacci) and $m = 3$ (Tribonacci) at stage of growth $S = 9$ and $S = 7$, respectively. The Fibonacci grating at stage $S = 9$ presents $N_{2,10} = 55$ zones with $N_{2,9} = 34$ transparent slits and the Tribonacci grating at stage $S = 7$ presents $N_{3,8} = 44$ zones with $N_{3,7} = 24$ transparent slits.

figure 1. In mathematical terms, the transmittance function $T_{m,S}(x)$ of an m-bonacci grating of order S is given by

$$T_{m,S}(x) = \sum_{l=1}^{N_{m,S+1}} t_{m,S,l}\ \mathrm{rect}\left(\frac{x - a(l - 1/2)}{a}\right). \tag{3}$$

3. Theoretical Fraunhofer diffraction patterns

Let us start by considering a one-dimensional diffraction grating described by a transmittance function $t(x)$. The Fraunhofer diffraction pattern is generated at the back focal plane of a lens placed against the grating when a plane wave impinges on the grating. Within the scalar approximation, and under monochromatic illumination, the focal irradiance is given by the squared modulus of the Fourier transform of the transmittance function [19]

$$I(x) = \left(\frac{A}{f}\right)^2 \left| \int_{-\infty}^{-\infty} t(x_0) \exp\left(-i\frac{2\pi x x_0}{f}\right) \mathrm{d}x_0 \right|^2, \tag{4}$$

where λ is the wavelength of the light, A is the amplitude of the incident plane wave, and f is the focal length of the lens. To determine the diffraction properties of an m-bonacci grating, we will obtain analytically its Fraunhofer pattern.

Replacing equation (4), the transmittance function $t(x)$ by $T_{m,S}(x)$ given by equation (3) and using the dimensionless transversal coordinate $u = \frac{ax}{f}$, equation (4) can be rewritten as

$$I_{m,S}(u) = \frac{1}{N_{m,S}^2} \left| \sum_{l=1}^{N_{m,S+1}} t_{m,S,j} \exp(-i2\pi ul) \right|^2 \mathrm{sinc}^2(u). \tag{5}$$

Note that the preceding equation has been normalized to have $I_{m,S}(0) = 1$.

By using equation (5), we have computed the Fraunhofer diffraction patterns provided by the Fibonacci and Tribonacci gratings represented in figure 1. The results are shown in

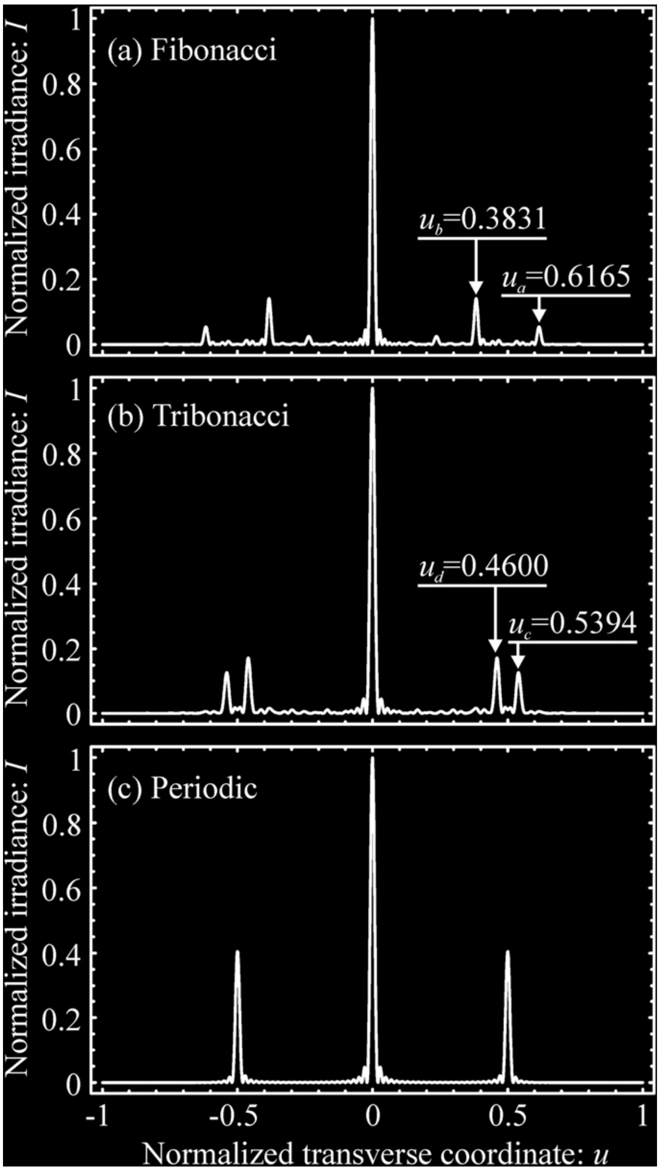

Figure 2. (a) Normalized Fraunhofer pattern irradiance versus the normalized transverse coordinate u for a Fibonacci grating ($S = 9$), (b) for a Tribonacci grating ($S = 7$), and (c) for a periodic grating (49 zones).

figure 2, together with the diffraction pattern corresponding to a conventional periodic grating with an intermediate number of zones (49 in our simulations), for comparison purposes. It can be seen that the main diffraction peaks in both cases coincide. However, the aperiodic distribution of zones according to the m-bonacci sequence produces a splitting of the first diffraction order in two irradiance peaks located at $u_a = 0.6165$ and $u_b = 0.3831$ for the Fibonacci grating; and at $u_c = 0.5394$ and $u_d = 0.46$ for the Tribonacci grating. Thus, the ratio of the transverse distances satisfies $u_a/u_b \approx \tau_2 = 1/(\varphi_2\text{-}1) = \varphi_2$ for the Fibonacci

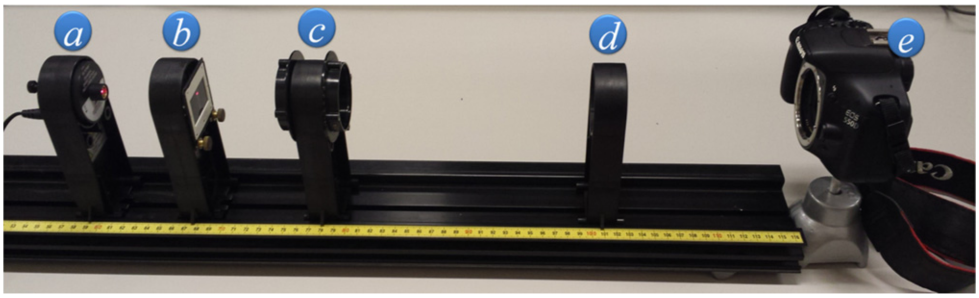

Figure 3. Experimental setup: (a) collimated laser diode; (b) diffraction grating; (c) variable density filter, consisting in a couple of linear polarizers (one rotatable); (d) thin lens; and (e) SLR (CMOS) sensor.

grating and $u_c/u_d \approx \tau_3 = 1/(\varphi_3\text{-}1)$ for the Tribonacci grating. In general, in an m-bonacci grating, the ratio of the position of the split first-order diffraction peaks tends to the ratio of transparent and opaque zones, $\tau_m = 1/(\varphi_m\text{-}1)$. We have verified that this property is also satisfied by m-bonacci gratings where the transparent and opaque zones have different width. In these cases, the position of the two diffraction peaks also tends to τ_m, but its relative maxima depend on the zones width. On the other hand, higher diffraction orders with a lower irradiance also appear (not shown in figure 2). However, the transverse distances between the local maxima corresponding to higher diffraction orders of the m-bonacci grating are not related through the generalized golden mean φ_m.

4. Classroom experiments

Interference and diffraction are an integral part of the subject photonic devices in the Bachelor's Degree in Industrial Electronics and Automation Engineering at the Technical University of Valencia, Spain. The theoretical background of this topic is supplemented with theoretical lessons in the classroom and with laboratory work. During the 2013–2014 academic year, the instructor introduced the concept of 'quasiperiodicity' as a property of ordered structures without translational periodicity. Quasiperiodic structures were presented in class in this way as a new motivating topic related to the Nobel Prize in Chemistry 2011 awarded to Dan Shechtman for the discovery of quasicrystals [20]. The Fibonacci and the here-presented m-bonacci sequences were the selected examples for one-dimensional quasiperiodic structures studied in class.

To experimentally check the diffraction properties of m-bonacci gratings, we registered the Fraunhofer diffraction patterns obtained when these quasiperiodic structures were illuminated with a monochromatic plane wave. The gratings were built by the students on graphic arts film using a photoplotter with 2400 lpi resolution [21]. Electronic engineering students were familiar with this technology because they use it to print photomasks for integrated circuit production. Three different gratings were fabricated, namely a Fibonacci grating with $S = 9$, a Tribonacci grating with $S = 7$, and a periodic grating with an intermediate number of periods (49 lines). The basic zone width was $a = 50\,\mu$m in all cases.

Figure 3 shows the setup used in our experiment. As a light source, a collimated laser diode ($\lambda = 650$ nm) was employed. The involved opto-mechanical elements were out-of-the-shelf components, available in any conventional teaching laboratory. The setup consisted of an optical bench on which the other items were conveniently placed. First, the laser source

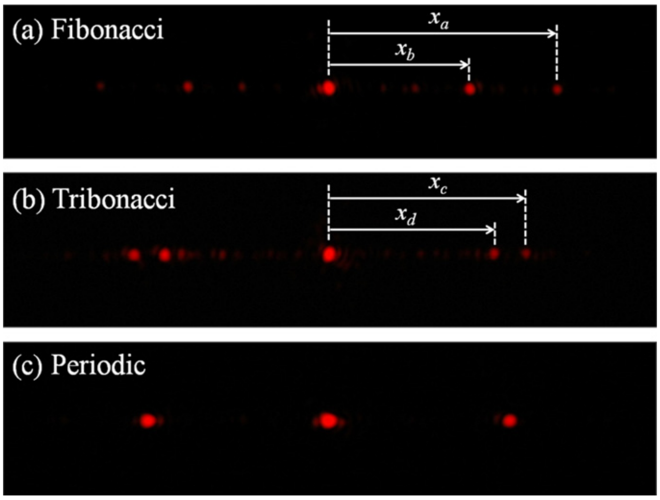

Figure 4. Experimental Fraunhofer diffraction patterns of a: (a) Fibonacci grating ($S = 9$); (b) Tribonacci grating ($S = 7$); and (c) conventional periodic grating (49 lines). The basic zone width was a = 50 μm in all three cases.

was adjusted to provide a plane wave illumination onto the object grating. Next, two linear polarizers controlled the irradiance level of the registered pattern by in-plane rotation of one respect to the other. Finally, at the back focal plane of a thin lens (focal length $f' = 200$ mm) a digital SLR sensor, 18 Mpx (22.3 × 14.9) mm, (Canon EOS550D camera) was used to register the Fraunhofer diffraction patterns.

The registered irradiance patterns corresponding to the three different gratings are shown in figure 4. These results are in full agreement with the theoretical predictions in figure 2. From figure 4(c) (periodic grating), the expected splitting of the first diffraction order is clearly seen in figures 4(a) and (b) (Fibonacci and Tribonacci gratings, respectively). Moreover, according to the theoretical predictions, the first diffraction orders of the m-bonacci gratings are obtained symmetrically at each side of the first-order diffraction of the periodic grating. Furthermore, from a quantitative point of view, it is easy to check that the ratios between their distances to the zero diffraction order agree with the relationship presented previously. In fact, distances in figure 4(a) (Fibonacci) are $x_a = 268$ pixels and $x_b = 166$ pixels, and their ratio is $x_a/x_b = 1.614$, which is in good agreement with the theoretical value $\tau_2 \approx 1/(\varphi_2\text{-}1) = \varphi_2 = 1.618$ for $S \to \infty$, being the discrepancy between the ratio x_a/x_b and τ_2 lower to 0.3%. For the Tribonacci case (figure 4(b)), we obtained $x_c = 237$ pixels and $x_d = 197$ pixels, so that $x_c/x_d = 1.203$. Again, this result is well matched by the predicted value $\tau_3 \approx 1/(\varphi_3\text{-}1) = 1.191$, being the discrepancy between the ratio x_c/x_d and τ_3 approximately 1%.

5. Conclusions

We have presented a simple optical experiment (from a didactical point of view) to analyse the Fraunhofer diffraction properties of m-bonacci gratings with the standard equipment available in most undergraduate physics laboratories. We have shown that the diffraction peaks generated by m-bonacci gratings are related to the generalized golden mean. In our opinion, the proposed experiment could be a motivational strategy to complete the topics of interference and diffraction of light waves in physics and engineering courses, and to introduce students to basic research activities. In fact, this experience has been implemented in experimental classes with successful results. An anonymous questionnaire given to electronic engineering students to assess the experience showed an overall high score of 8.9 (in a 1 to 10 scale, where 10 is the highest score). The students thought it was very interesting and motivating (scoring 9.2), but with a slight mathematical difficulty (scoring 7.8). On the other hand, the experiment helped students to understand the basic principles of quasiperiodic structures (scoring 8.7). It is worth mentioning that, with this experimental methodology, other aperiodic structures such as the Thue-Morse gratings can be also studied.

Acknowledgments

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