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Unisolvency for multivariate polynomial interpolation in Coatmèlec configurations of nodes

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ABSTRACT

A new and straightforward proof of the unisolvability of the problem of multivariate polynomial interpolation based on Coatmèlec configurations of nodes, a class of properly posed set of nodes defined by hyperplanes, is presented. The proof generalizes a previous one for the bivariate case and is based on a recursive reduction of the problem to simpler ones following the so-called Radon–Bézout process.

1. Introduction

The problem of polynomial interpolation of one-dimensional data has a widely known solution. However, despite its apparent simplicity, multivariate polynomial interpolation remains a topic of current research [1–3]. The existence and uniqueness of the interpolation polynomial strongly depends on the geometrical distribution of the interpolation points. The distribution of points for which the interpolation problem is unisolvable is referred to as properly posed set of nodes (PPSN).

The mathematical characterization of the most general PPSN is not currently known. The configurations of nodes based on algebraic varieties, such as those of Bos [4] and Liang et al. [5,6], are very general but non-constructive. In a computational setting, configurations based on hyperplanes, such as those of Coatmèlec [7] and Chung and Yao [8], are preferred.

Surprisingly, the configuration of nodes introduced by Coatmèlec [7] in the plane has received several names: DH-set [2], straight line type node configuration [5], PPSN with node configuration A [9], straight line type node configuration A [10], PPSN by the recursive construction theorem using lines [11], and PPSN by line-superposition process [12].

In this paper, a new proof of the unisolvability of the interpolation problem for Coatmèlec configuration of nodes in arbitrary dimensions is presented. The proof is based on a Bézout–Radon process [13,14]. Chui and Lai [9] present a proof for the bivariate case only, state the result in arbitrary dimension, but did not prove it because of complications in their notation. Multidimensional interpolation is the basis to develop different numerical methods. The results of this paper permit to design, for example, generalized finite difference methods in irregular meshes based on Coatmèlec configuration of nodes in two [15] or more dimensions.

The contents of this paper are as follows. The definitions and notation required to set our main theorem are presented in the next section. The proof of this theorem is detailed in Section 3. Finally, in the last section, the main conclusions are summarized.

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2. Presentation of the problem

Let $\Pi_m(\mathbb{R}^k)$ be the vector space of multivariate polynomials of degree not greater than m with k variables. Let $w=(x_1,\ldots,x_k)^\top\in\mathbb{R}^k$, where $^\top$ denotes transpose, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $j=(j_1,\ldots,j_k)^\top\in\Gamma:=\mathbb{N}_0^k$, $|j|=j_1+\cdots+j_k$, $w^j=x_1^{j_1}x_2^{j_2}\cdots x_k^{j_k}$, and $\Gamma_m:=\{j\in\Gamma:|j|\leqslant m\}$. The set of multivariate monomials $\{w^j\}_{j\in\Gamma_m}$ is a basis of $\Pi_m(\mathbb{R}^k)$, i.e., every polynomial $p_m(w)$ may be written uniquely as $\sum_{j\in\Gamma_m}a_jw^j$, with $a_j\in\mathbb{R}$. Hence, the vector space $\Pi_m(\mathbb{R}^k)$ has dimension $N=C^k_{k+m}$, where C^k_n is the binomial coefficient $\binom{n}{k}$.

Let $\Gamma^s:=\{j\in\Gamma_m:|j|=s\}$, $s=0,1,\ldots,m$. Note that $\Gamma_m=\cup_{s=0}^m\Gamma^s$, the cardinal $\#\Gamma^s=C^{k-1}_{k-1+s}$, and $\#\Gamma_m=\sum_{s=0}^mC^{k-1}_{k-1+s}=N$. The set of sth degree monomials may be represented as a column vector of length $\#\Gamma^s$ given by $w^{(s)}:=(x_1^s,x_1^{s-1}x_2^1,\ldots,x_{i_1}x_{i_2}\cdots x_{i_s},\ldots,\ x_{k-1}^1x_k^{s-1},x_k^s)^\top$, for all $i=(i_1,\ldots,i_s)^\top\in\mathbb{N}_0^s$, and $1\leqslant\ i_1\leqslant i_2\leqslant\cdots\leqslant i_s\leqslant k$. Note that $w^{(0)}=(1)\in\mathbb{R}$, $w^{(1)}=w\in\mathbb{R}^k$, and each component of the vector $w^{(s)}$ corresponds to a unique monomial w^j with $j\in\Gamma^s$. Using this notation, every polynomial $p_m(w)\in\Pi_m(\mathbb{R}^k)$ may be written as $\sum_{s=0}^m\sum_{j\in\Gamma^s}a_jw^j$.

Here on, a node refers to a point in $\mathbb{R}^k$ and a configuration of nodes (CN) is a set of pairwise distinct nodes $X_m=\{w_i\}_{i=1}^N$ where $w_i\equiv(x_{(1,i)},x_{(2,i)},\ldots,x_{(k,i)})^\top\in\mathbb{R}^k$.

The Lagrange interpolation problem may be stated as follows: Given a CN X_m and an arbitrary set of real numbers $\{f_i\in\mathbb{R}\}_{i=1}^N$, find a polynomial $p_m(w)\in\Pi_m(\mathbb{R}^k)$ such that

$$p_m(w_i):=\sum_{j\in\Gamma_m}a_jw_i^j=f_i,\quad i=1,2,\ldots,N. \tag{1}$$

This problem is *properly posed* with respect to X_m if it has a unique solution (unisolvability) for every set $\{f_i\}_{i=1}^N$. Compared with the one-dimensional case where the solvability is always assured, the solvability of multivariate interpolation depends strongly on the geometrical distribution of the nodes. A CN X_m is said to be a *properly posed set of nodes* (PPSN) if the Lagrange interpolation problem is properly posed with respect to X_m.

Eq. (1) is a system of N linear equations with a multivariate Vandermonde matrix V_m, i.e., $(V_m)_{ij}=w_i^j$, where $j\in\Gamma_m$, $w_i\in X_m$, and $1\leqslant i\leqslant N$. Note that this matrix looks a little bit bizarre since rows and columns are indexed by different structural entities. A graded lexicographical order in the set of multiindices Γ_m may be introduced to enhance the notation (see Ref. [16]) but this is not required in this paper.

The following theorem summarizes some previously known results.

Theorem 1. *Let $X_m=\{w_i\}_{i=1}^N$ be a CN in k dimensions and V_m the corresponding multivariate Vandermonde matrix, then the following expressions are equivalent:*

(i) *X_m is a PPSN in $\mathbb{R}^k$.*
(ii) *V_m is a nonsingular matrix, i.e., det$(V_m)\neq0$.*
(iii) *rank$(V_m)=N$.*

Let $X_m\equiv X_{(m,k)}=\{w_i\}_{i=1}^N\subset\mathbb{R}^k$ be a CN with $N=C^k_{m+k}$ nodes in k dimensions. Let us define by induction on k the following CNs, first introduced by Coatmèlec [7,9].

Definition 2. A CN $X_m\equiv X_{(m,k)}\subset\mathbb{R}^k$ is Coatmèlec in k dimensions if $X_{(m,k)}=\bigcup_{p=0}^mX_{(p,k-1)}$ with $\#X_{(p,k-1)}=C^{k-1}_{p+k-1}$ and there exists $m+1$ hyperplanes $\gamma_0,\gamma_1,\ldots,\gamma_m$ such that $X_{(m,k-1)}\subset\gamma_m$ and $X_{(p,k-1)}\subset\gamma_p\setminus\bigcup_{q=p+1}^m\gamma_q$, for $0\leqslant p\leqslant m-1$, with each $X_{(p,k-1)}$ being Coatmèlec in $(k-1)$ dimensions by identifying each hyperplane γ_p with $\mathbb{R}^{k-1}$.

Note that, in one dimension, every CN $X_m\equiv X_{(m,1)}\subset\mathbb{R}$ is Coatmèlec because all its nodes are pairwise distinct, i.e., $w_i\neq w_j$, if $i\neq j$. Note also that, in Definition 2, only one node belongs to the hyperplane γ_m.

The main result of this paper is a proof of the following theorem.

Theorem 3. *Every Coatmèlec CN X_m in k dimensions is a properly posed set of nodes in $\mathbb{R}^k$.*

3. Proof of the main theorem

Our proof makes use of the following lemmas.

Lemma 4. *Let us take the CN X_m where the nodes $\{w_i\}_{i=1}^N$ are represented as column vectors in $\mathbb{R}^k$, and the CN $\widehat{X}_m$ whose nodes are $\hat{w}_i=w_0+Hw_i$, $i=1,\ldots,N$, where w_0 is an arbitrary vector and H is a non-singular matrix of dimension k. Let V_m and $\widehat{V}_m$ be the Vandermonde matrices associated to the CNs X_m and $\widehat{X}_m$, respectively. If rank$(V_m)=N$, then rank$(\widehat{V}_m)=N$.*

Proof of Lemma 4. For every set of real numbers $\{f_i \in \mathbb{R}\}_{i=1}^N$, there exists one and only one interpolating polynomial such that $\hat{p}_m(\hat{w}_i) = f_i$, given by $\hat{p}_m(\hat{x}) = p_m(H^{-1}(x - w_0))$ where $p_m(x)$ is the unique interpolating polynomial for X_m given by Theorem 1. Therefore, $\text{rank}(\widehat{V}_m) = N$.

Lemma 5. *Let $\{\hat{x}_i : i = 1, \ldots, k\}$ be an orthonormal basis of $\mathbb{R}^k$, and n_1 an arbitrary vector. There always exists an orthogonal matrix H, representing a rotation in $\mathbb{R}^k$, which transform the vector $\hat{x}_1$ onto $H\hat{x}_1 = \hat{n}_1 = n_1/\|n_1\|$.*

Proof of Lemma 5. If $\hat{n}_1 = \hat{x}_1$, then $H = I$, the identity matrix. Otherwise, let us apply the procedure of Gram–Schmidt orthonormalization to vectors $\{\hat{x}_1, n_1\}$, yielding

$$\hat{q}_1 = \hat{x}_1, \quad q_2 = n_1 - (n_1 \cdot \hat{q}_1)\hat{q}_1, \quad \hat{q}_2 = \frac{q_2}{\sqrt{q_2 \cdot q_2}} = \frac{q_2}{\|q_2\|},$$

where the dot is the ordinary Euclidean dot product. An arbitrary vector q can be written as $q = q_\perp + q_{\text{Vert}}$, where $q_\| = (q \cdot \hat{q}_1)\hat{q}_1 + (q \cdot \hat{q}_2)\hat{q}_1 = QQ^\top q$, where $Q = [\hat{q}_1; \hat{q}_2]$ is the rectangular matrix whose columns are the vectors $\hat{q}_i$; note that $Q^\top Q$ is the identity matrix of dimension 2. Taking the vector $q_\perp = q - q_\|$ as the rotation axis for the rotation matrix H results in $Hq = q_\perp + Hq_\| = (I - QQ^\top)q + QRQ^\top q$, where R is the standard two-dimensional rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \cos\theta = \hat{x}_1 \cdot \hat{n}_1, \quad \sin\theta = \sqrt{1 - (\hat{x}_1 \cdot \hat{n}_1)^2}.$$

Hence, $H = I - QQ^\top + QRQ^\top$ is a rotation matrix ($HH^\top = H^\top H = I$ and $\det(H) = 1$) such that $H\hat{x}_1 = \hat{n}_1$.

Proof of Theorem 3. Let us use the induction principle over m and k. Let us first consider $m = 0$ and any $k \in \mathbb{N}$. Clearly $X_0 = w_1$ and $\text{rank}(V_0) = 1 = N$. We consider next $k = 1$ and $m \neq 0$. The corresponding CN is Coatmèlec in one dimension and the coefficient matrix is a (one-dimensional) Vandermonde matrix with maximal rank $C^1_{m+1} = m + 1 = N$, since the nodes are pairwise distinct.

By the induction hypothesis, let us assume that the theorem holds for either $m - 1$ or $k - 1$, and let us prove that it holds for m and k. Here on, let us take $n = m + k$. Since X_m is a Coatmèlec CN in k dimensions, the following conditions are fulfilled

$$\begin{aligned}
X_{(m,k-1)} &= \left\{w_1, w_2, \ldots, w_{C^{k-1}_{n-1}}\right\} \subset \gamma_m, \\
X_{(m-1,k-1)} &= \left\{w_{C^{k-1}_{n-1}+1}, \ldots, w_{C^{k-1}_{n-1}+C^{k-1}_{n-2}}\right\} \subset \gamma_{m-1} \setminus \gamma_m, \\
X_{(m-2,k-1)} &= \left\{w_{C^{k-1}_{n-1}+C^{k-1}_{n-2}+1}, \ldots, w_{C^{k-1}_{n-1}+C^{k-1}_{n-2}+C^{k-1}_{n-3}}\right\} \subset \gamma_{m-2} \setminus \gamma_{m-1} \cup \gamma_m, \\
&\;\;\vdots \\
X_{(0,k-1)} &= \{w_N\} \subset \gamma_0 \setminus \gamma_1 \cup \cdots \cup \gamma_m,
\end{aligned}$$

where

$$X_m = X_{(m,k-1)} \cup X_{(m-1,k-1)} \cup \cdots \cup X_{(0,k-1)}.$$

The multivariate Vandermonde matrix associated to the Lagrange interpolation problem in the CN X_m may be written as

$$V_m = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ w_1^{(1)} & w_2^{(1)} & \cdots & w_{C_n^k}^{(1)} \\ w_1^{(2)} & w_2^{(2)} & \cdots & w_{C_n^k}^{(2)} \\ \vdots & \vdots & & \vdots \\ w_1^{(m)} & w_2^{(m)} & \cdots & w_{C_n^k}^{(m)} \end{pmatrix}.$$

Let us apply the affine transformation $\hat{w} = w_0 + Hw$ to all the nodes of the CN, where H is the orthogonal matrix given in Lemma 5, that transforms the x_k coordinate axis in $\mathbb{R}^k$ into the normal vector to the hyperplane γ_m, and w_0 is the distance between the intersection point of the (new) rotated x_k axis and the hyperplane γ_m.

The application of the affine transformation nullifies the kth coordinates of the vectors $\{\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{C^{k-1}_{n-1}}\}$, hence $\hat{w}_i = (\hat{x}_{(1,i)}, \hat{x}_{(2,i)}, \ldots, \hat{x}_{(k-1,i)}, 0)^\top$. Let $\widehat{V}_m$, where $(\widehat{V}_m)_{ij} = \hat{w}_i^j$, be the coefficient matrix of the transformed linear system of equations. From Lemma 4, $\text{rank}(V_m) = \text{rank}(\widehat{V}_m)$.

The rows and columns of the matrix $\widehat{V}_m$ may be sorted by renaming the nodes $\hat{w}_i$ to $\tilde{w}_i$, in order to group all its zero elements into its left-bottom part. This process preserves the rank. The resulting matrix $\widetilde{V}_m$ has the following structure

$$\begin{pmatrix} A & B \\ 0 & A'D \end{pmatrix}, \tag{2}$$

where A is the $C_{n-1}^{k-1} \times C_{n-1}^{k-1}$ matrix given by

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tilde{w}_1^{(1)} & \tilde{w}_2^{(1)} & \cdots & \tilde{w}_{C_{n-1}^{k-1}}^{(1)} \\ \tilde{w}_1^{(2)} & \tilde{w}_2^{(2)} & \cdots & \tilde{w}_{C_{n-1}^{k-1}}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_1^{(m)} & \tilde{w}_2^{(m)} & \cdots & \tilde{w}_{C_{n-1}^{k-1}}^{(m)} \end{pmatrix},$$

B is the $C_{n-1}^{k-1} \times C_{n-1}^{k}$ matrix

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tilde{w}_{C_{n-1}^{k-1}+1}^{(1)} & \tilde{w}_{C_{n-1}^{k-1}+2}^{(1)} & \cdots & \tilde{w}_{C_n^k}^{(1)} \\ \tilde{w}_{C_{n-1}^{k-1}+1}^{(2)} & \tilde{w}_{C_{n-1}^{k-1}+2}^{(2)} & \cdots & \tilde{w}_{C_n^k}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{C_{n-1}^{k-1}+1}^{(m)} & \tilde{w}_{C_{n-1}^{k-1}+2}^{(m)} & \cdots & \tilde{w}_{C_n^k}^{(m)} \end{pmatrix},$$

D is the $C_{n-1}^{k} \times C_{n-1}^{k}$ diagonal matrix

$$D = \begin{pmatrix} \hat{x}_{(k,C_{n-1}^{k-1}+1)} & 0 & \cdots & 0 \\ 0 & \hat{x}_{(k,C_{n-1}^{k-1}+2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{x}_{(k,C_n^k)} \end{pmatrix},$$

A' is the $C_{n-1}^{k} \times C_{n-1}^{k}$ matrix given by

$$A' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tilde{w}_{C_{n-1}^{k-1}+1}^{(1)} & \tilde{w}_{C_{n-1}^{k-1}+2}^{(1)} & \cdots & \tilde{w}_{C_n^k}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{C_{n-1}^{k-1}+1}^{(m)} & \tilde{w}_{C_{n-1}^{k-1}+2}^{(m)} & \cdots & \tilde{w}_{C_n^k}^{(m)} \end{pmatrix},$$

and finally 0, cf. Eq. (2), represents the null matrix of dimensions $C_{n-1}^{k} \times C_{n-1}^{k-1}$. We recall that $C_n^k = C_{n-1}^{k-1} + C_{n-1}^{k}$.

The square matrix A is a multivariate Vandermonde matrix in $(k-1)$ variables and the C_{n-1}^{k-1} nodes $\{\tilde{w}_i\}$ are a Coatmèlec CN in $(k-1)$ dimensions. Therefore, by the induction hypothesis, $\operatorname{rank}(A) = C_{n-1}^{k-1}$.

The diagonal matrix D is nonsingular, i.e., $\hat{x}_{k,i} \neq 0$, for $i = C_{n-1}^{k-1}+1, \ldots, C_n^k$, because if there existed at least an i with $\hat{x}_{k,i} = 0$, then there would be at least $C_{n-1}^{k-1}+1$ different nodes lying in the hyperplane γ_m, but this is not possible because X_m is a Coatmèlec CN. Hence, $\operatorname{rank}(A'D) = \operatorname{rank}(A')$. Moreover, the matrix A' is also a multivariate Vandermonde matrix corresponding to the C_{n-1}^{k} nodes that do not belong to the hyperplane γ_m. Since the Coatmèlec property of a CN does not change under either rotation or translation of all the nodes, the CN $\{\tilde{w}_i\}$, $i = C_{n-1}^{k-1}+1, \ldots, C_n^k$, is also a Coatmèlec CN. The induction hypothesis yields that the rank of matrix A' is C_{n-1}^{k}.

Finally, the rank of the $C_n^k \times C_n^k$ matrix $\widetilde{V}_m$ is $\operatorname{rank}(A) + \operatorname{rank}(A') = C_{n-1}^{k-1} + C_{n-1}^{k} = C_n^k$, and the theorem is proved.

4. Conclusions

The unisolvency of the problem of multivariate polynomial interpolation in a Coatmèlec CN, a kind of properly posed set of nodes defined by hyperplanes, has been shown through a new and straightforward proof. This proof uses elementary techniques from linear algebra. This fact permits the understanding of the topic by nonexperts and opens the possibility of it being incorporated in numerical analysis textbooks.

The geometrical condition characterizing Coatmèlec CNs is one of the most general conditions currently available for the characterization of properly posed set of nodes defined by hperplanes, which is easier and more efficient to be checked by an automatic computational software than the widely known geometrical characterization of Chung and Yao [8]. Therefore, Coatmèlec CNs are useful in mesh generation for the numerical solution of partial differential equations in irregular domains, such as generalized finite difference methods.

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