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# Local Isoperimetric Inequalities for Sectors on Surfaces and Cones

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**Abstract** On surfaces we give conditions under which the solution of a restricted local isoperimetric problem for sectors with small solid angle is the circular sector and we characterize these surfaces. Also we study this problem for general spherical cones on hypersurfaces in higher dimensional Riemannian manifolds.

**Keywords** isoperimetric, cones, solid angle
**MR(2000) Subject Classification** 53B20

## 1 Introduction

Let $M^2$ be a $C^\infty$ Riemannian surface (manifold of dimension two), let $p \in M$ and let $u, v \in T_pM$ be unit vectors. Let $\gamma_u$ and $\gamma_v$ be the geodesics starting from $p$ with $\gamma_u{}'(0) = u$ and $\gamma_v{}'(0) = v$. Let $\Gamma$ be a $C^\infty$ curve joining two points of $\gamma_u$ and $\gamma_v$ respectively, without self-intersections. The set of points of $M$ contained between the geodesics and the curve is called a sector with vertex $p$, base $\Gamma$ and sides $\gamma_u$ and $\gamma_v$. We define the angle of this sector as $\angle(u, v)$ and the direction as $\frac{u+v}{2}$. We only consider sectors that are closed, simply connected and bounded.

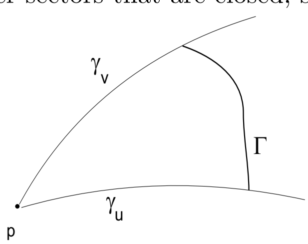


Fig. 1 Sector of vertex $p$ and base $\Gamma$

*The isoperimetric problem for sectors consists in finding the sector with minimal base length, for fixed vertex, direction, angle and area.*

Bandle [1, 2] showed that in the plane $\mathbb{R}^2$ and for angles lower than $\pi$ the solution of the isoperimetric problem is the circular sector (a sector where the base is a piece of the geodesic circle with centre being the vertex of the sector). This result is valid also for sectors in the sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. Bahn and Hong [3] proved the following generalization. Let $M$ be a surface with curvature $K$ bounded from above by a constant $C$, let $D$ be a sector

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on $M$ with base $\Gamma$ and angle less than $\pi$, let $\overline{D}$ be a circular sector with base $\overline{\Gamma}$ on the surface of constant curvature $C$ with the same area as $D$. Then the length of $\Gamma$ is greater than or equal to the length of $\overline{\Gamma}$, and equality holds only when $D$ is isometric to $\overline{D}$.

Similar isoperimetric inequalities on Lorentzian surfaces are obtained by these authors [4, 5].

In this paper, we try to answer the question: For which surfaces $M$ is the solution of the isoperimetric problem for sectors a circular sector? In order to state the principal results we need the following definitions. Let $\omega$ be the Riemannian volume element of $M$ and let $du$ be arclength $S^1 \subset T_pM$. We shall denote by $\theta(t,u)$ the real function defined by

$$\omega(\exp_p tu) = t\theta(t,u)dtdu.$$

This function is called the infinitesimal change of volume function with respect to $p$ in [6] (see §3.2 for more details). The function $\theta(t,u)$ is called radial if it depends only on $t$. Observe that $M$ is spherically symmetric with respect to $p$ if and only if the infinitesimal change of volume function with respect to $p$ is radial.

A curve $\Gamma$ satisfying $\Gamma \cap \mathrm{cut}(p) = \emptyset$, where $\mathrm{cut}(p)$ denotes the cut locus of $p$, is said to be *radially connected to a point* $p$ if for each $q \in \Gamma$ there is a minimizing geodesic segment $\eta$ joining $p$ and $q$ with $\eta \cap \Gamma = \{q\}$.

We shall give a partial answer to our question in the category of radially connected curves. First we shall prove, under the hypothesis of spherical symmetry around the vertex of the sector, the following theorem:

**Theorem 1.1** *Suppose that $\theta(t,u)$ is radial. Let $q \in M - (\mathrm{cut}(p) \cup \{p\})$. Let $\Gamma$ be a curve in $M$ radially connected to $p$. Let $D_\alpha$ be the sector of vertex $p$, with one side being the geodesic segment joining $p$ and $q$, angle $\alpha$ and base being the corresponding piece of $\Gamma$. Then there is a $\delta > 0$ such that for every angle $\alpha$ less than $\delta$ there is a circular sector with the same vertex, angle and area as $D_\alpha$, one side being contained in the geodesic segment joining $p$ and $q$ and with base of smaller or equal length, with equality only when $D_\alpha$ is a circular sector with vertex at $p$.*

As a consequence we have a uniqueness theorem under the same spherical symmetry hypothesis:

**Theorem 1.2** *Suppose that $\theta(t,u)$ is radial. Let $q$ be a point in $\in M - (\mathrm{cut}(p) \cup \{p\})$. Let $\Gamma$ be a curve in $M$ radially connected to $p$ and containing $q$. Suppose that for some $\alpha > 0$, all sectors of vertex $p$, angle less than $\alpha$, base being a piece of $\Gamma$ and one side being the geodesic joining $p$ and $q$ are solutions of the isoperimetric problem. Then in a neighborhood of $q$, $\Gamma$ is a piece of a geodesic circle of centre $p$.*

The next theorem and corollary show that spherical symmetry is a necessary hypothesis for the above theorem when considering the problem for sectors with side and angle given.

**Theorem 1.3** *Suppose $\theta(t,u)$ is not radial. Then there exist a point $q \in M - (\mathrm{cut}(p) \cup \{p\})$ and a curve $\Gamma$ containing $q$ such that the length of each base (on $\Gamma$) of the sector of vertex $p$, one of the sides being the geodesic joining $p$ and $q$ and angle $r$, is smaller than the length of the base of the appropriate circular sector with the same area.*

As a consequence, the reverse of Theorem 1.2 is also true in some way:

**Corollary 1.4** *Suppose that, for a fixed vertex $p$, the solution of the local isoperimetric problem for sectors of small angle is the circular sector. Then the infinitesimal change of volume function is radial.*

Observe that the sector with vertex $p$ and base $\Gamma$ is the cone $p \lhd \Gamma$, that is, the union of the geodesic segments from $p$ to points of $\Gamma$. Isoperimetric inequalities for cones in large dimensions are obtained by several authors [7–10]

We will provide an alternative version of Theorem 1.2 which holds in general dimensions, without assuming $\theta(t,u)$ radial. We need some terminology, partially taken from ([11]).

An embedded hypersurface $\Omega$ in $M$ satisfying $\Omega \cap \text{cut}(p) = \emptyset$ is said to be radially connected to a point $p$ if for each $q \in \Omega$ there is a minimizing geodesic segment $\eta$ joining $p$ and $q$ with $\eta \cap \Omega = \{q\}$. From this definition it follows that if an embedded hypersurface $\Omega$ is radially connected to a point $p$, then it is contained in $M - (\text{cut}(p) \cup \{p\})$.

Let $M$ be a Riemannian $C^\infty$ manifold of dimension $n$, $p \in M$ and let $\Omega$ be an embedded $C^\infty$ hypersurface radially connected to $p$. Given $q \in \Omega$, $a$ will denote the unit vector in

$T_pM$ satisfying $q = \exp_p \text{dist}(p,q)a$, that is $a = \frac{\exp_p{}^{-1} q}{\|\exp_p{}^{-1} q\|}$.

By $B_r(a)$ we shall denote the geodesic ball of centre $a$ and radius $r$ in the unit sphere $S^{n-1} \subset T_pM$. We shall only consider $r$ such that $\exp_p(tu)$ is a minimizing geodesic from $p$ to some point in $\Omega$ for $u \in B_r(a)$. For every $u \in B_r(a)$ we define

$$l(u) = \text{minimum}\{t \in ]0, c(u)[: \ \exp_p tu \in \Omega\}.$$

From these definitions it follows that $\gamma_u(l(u)) \in \Omega$ and $l(a) = \text{dist}(p,q)$.

**Definition 1.5** $\Omega(q,r) := \{\exp_p l(u)u: \ u \in B_r(a)\}$

$C\Omega(q,r) := p \triangleleft \Omega(q,r) = \{\gamma_u([0, l(u)]): \ u \in B_r(a)\}$

*$C\Omega(q,r)$ is called the cone in $M$ over $\Omega$ with vertex $p$, centre $q$ and angle $2r$. $\Omega(q,r)$ will be called the base of this cone and $a$ will be called the direction of the cone* (see Fig. 2).

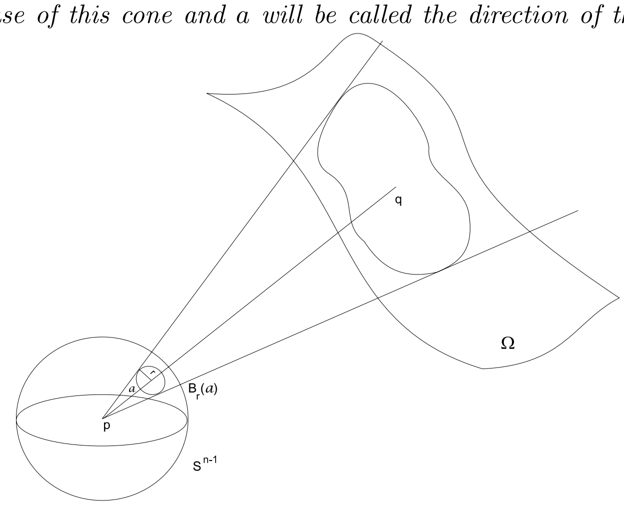


Fig. 2 Cone with vertex $p$, solid angle $2r$, centre $q$ and base $\Omega(q,r)$

Observe that $C\Omega(q,r)$ is contained in a normal neighborhood of $p$.

The sectors studied above are just the cones over $\Omega = \Gamma$ with vertex $p$ and angle $2r = \angle(u,v)$ and base $\Gamma = \Omega(q,r)$ where $q = \exp_p l\left(\frac{u+v}{2}\right)\frac{u+v}{2}$.

If $\Omega = S_R(p)$, the geodesic sphere of radius $R$ and centre $p$ with $R > 0$, then $C\Omega(q,r)$ is called a *spherical cone with solid angle* $2r$, *centre* $q$ *and radius* $R$.

*The isoperimetric problem for cones consists in finding the cone with minimal volume of its base for fixed vertex, solid angle and volume of the cone.*

We prove the following result:

**Theorem 1.6** *Let $\Omega$ be a hypersurface of $M$ radially connected to $p$. Suppose that for some $\alpha > 0$, all the cones of vertex $p$, angle less than $\alpha$ and base being a piece of $\Omega$ are solutions of the isoperimetric problem. Then $\Gamma$ is a piece of a geodesic sphere of centre $p$.*

This result shows that that to be a solution of isoperimetric problem for cones of small solid angles and every direction from a fixed point is very different from that of being a solution of isoperimetric problem, because there are known spaces (for instance, the Euclidean plane with a small smooth bump at $(1/2, 0)$) where the solution of the isoperimetric problem is not a geodesic sphere.

The hypotheses of Theorem 1.6 cannot be weakened to that the cones of small solid angles in a fixed direction are the solutions of the isoperimetric problem at least without adding a new condition, as Theorem 1.3 shows. At this moment we do not know if the condition to be added is spherical symmetry or something weaker.

The obstacle to generalizing Theorem 1.1 is the lack of formula (5) for higher dimensions.

We prove these results by Taylor series computations.

## 2 Proofs of Theorems 1.1 and 1.3

In this section, $M$ will denote a surface, $p \in M$, $\Gamma$ will denote a one-dimensional embedded submanifold radially connected to $p$, and $q \in \Gamma$. Moreover, $\theta(t,u)$ will denote the infinitesimal change of volume function of $M$ respect to $p$. Let $p$, $q$ and $\Gamma$ be as in Theorem 1.1. Let $a = \frac{{\exp_p}^{-1} q}{\|{\exp_p}^{-1} q\|} \in S^1 \subset T_pM$. Let $\{a, v\}$ be an orthonormal base of $T_pM$. Let $\varphi(s) = a\cos s + v \sin s \in S^1 \subset T_pM$ for $s \in [0, 2\pi]$. We denote by $D(q,\alpha)$ and $\Gamma(q,\alpha)$ the sector of vertex $p$, angle $\alpha$, base being a piece of $\Gamma$ and one side being the geodesic joining $p$ and $q$, and its base, respectively. For $0 < R < \mathrm{d}(p, \mathrm{cut}(p))$ let $\overline{q} = \exp_p Ra$. We denote by $D_R(\overline{q},\alpha)$ and $S_R(\overline{q},\alpha)$ the circular sector of vertex $p$, angle $\alpha$, radius $R$ and one of the sides being the geodesic joining $p$ and $\overline{q}$, and it base respectively. By using geodesic polar coordinates and the notation $A(t,s) = t\theta(t,s)$, $l(s) = l(\varphi(s)) = \text{minimum}\{t \in ]0,\infty[ : \exp_p tu \in \Gamma\}$ and $\nu_{\varphi(s)}$ being the unit normal vector to $\Gamma$ at $\exp_p l(s)\varphi(s)$, it is easy to show that

$$\text{area}(D(q,\alpha)) = \int_0^\alpha \int_0^{l(s)} A(t,s)dtds, \tag{1}$$

$$\text{length}(\Gamma(q,\alpha)) = \int_0^\alpha \frac{A(l(s),s)}{|\langle {\gamma_{\varphi(s)}}'(l(s)), \nu_{\varphi(s)}\rangle|} ds, \tag{2}$$

$$\text{area}(D_R(\overline{q},\alpha)) = \int_0^\alpha \int_0^R A(t,s)dtds, \tag{3}$$

$$\text{length}(S_R(\overline{q},\alpha)) = \int_0^\alpha A(R,s)ds. \tag{4}$$

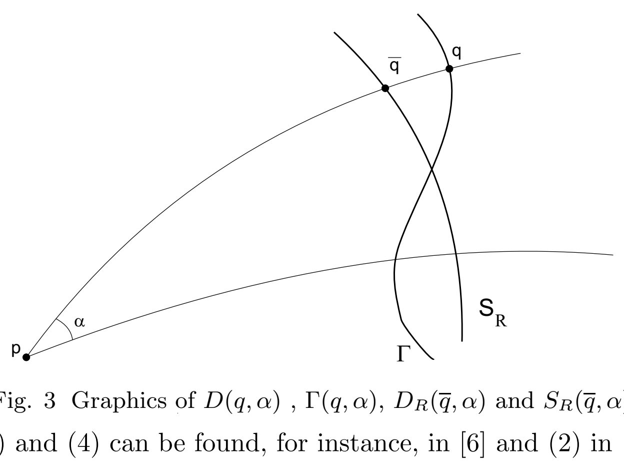


Fig. 3 Graphics of $D(q,\alpha)$ , $\Gamma(q,\alpha)$, $D_R(\overline{q},\alpha)$ and $S_R(\overline{q},\alpha)$

Formulae (1), (3) and (4) can be found, for instance, in [6] and (2) in [12] and [11].

To compute $|\langle {\gamma_{\varphi(s)}}'(l(s)), \nu_{\varphi(s)}\rangle|$ we proceed as follows. Let $\beta(s) = \exp_p l(s)\varphi(s)$ be a parametrization of $\Gamma$. Then $\left\{\frac{\beta'(s)}{|\beta'(s)|}, \nu_{\varphi(s)}\right\}$ is an orthonormal base of $T_{\beta(s)}M$. Hence

$$1 = \langle \gamma'_{\varphi(s)}(l(s)), \gamma'_{\varphi(s)}(l(s))\rangle = \left\langle \gamma'_{\varphi(s)}(l(s)), \frac{\beta'(s)}{|\beta'(s)|}\right\rangle^2 + \langle \gamma'_{\varphi(s)}(l(s)), \nu_{\varphi(s)}\rangle^2.$$

Furthermore, if $x(s,t) = \exp_p t\varphi(s)$, then $\gamma'_{\varphi(s)}(l(s)) = \partial_t x(s, l(s))$ and $\beta'(s) = \partial_s x(s, l(s)) +$

$\partial_t x(s,l(s))l'(s)$. Therefore

$$\begin{aligned}\langle\gamma'_{\varphi(s)}(l(s)),\nu_{\varphi(s)}\rangle^2 &= 1-\left\langle\gamma'_{\varphi(s)}(l(s)),\frac{\beta'(s)}{|\beta'(s)|}\right\rangle^2\\ &= 1-\frac{\langle\partial_t x,\partial_s x+\partial_t x l'\rangle^2}{\langle\partial_s x+\partial_t x l',\partial_s x+\partial_t x l'\rangle^2}\\ &= 1-\frac{(\langle\partial_t x,\partial_s x\rangle+|\partial_t x|^2 l')^2}{|\partial_s x|^2+2\langle\partial_t x,\partial_s x\rangle l'+|\partial_t x|^2 l'^2}.\end{aligned}$$

Using that $|\partial_t x|=1$ and $\langle\partial_t x,\partial_s x\rangle=0$ (by the Gauss lemma) we get

$$\langle\gamma'_{\varphi(s)}(l(s)),\nu_{\varphi(s)}\rangle^2=1-\frac{l'^2}{|\partial_s x|^2+l'^2}=\frac{|\partial_s x|^2}{|\partial_s x|^2+l'^2}.$$

But $A(t,s)=t\theta(t,\varphi(s))\ dt\ ds=w(\exp_p t\varphi(s))=w(x(s,t))$ (where $w$ be the Riemannian volume element of $M$), then

$$t\theta(t,\varphi(s))=w(\partial_t x,\partial_s x)=|\partial_s x|w\left(\partial_t x,\frac{\partial_s x}{|\partial_s x|}\right)=|\partial_s x|,$$

$$|\langle\gamma'_{\varphi(s)}(l(s)),\nu_{\varphi(s)}\rangle|=\frac{A(l(s),s)}{\sqrt{A(l(s),s)^2+l'(s)^2}}. \tag{5}$$

By substitution in (2), we obtain

$$\text{length}(\Gamma(q,\alpha)) = \int_0^\alpha\sqrt{A(l(s),s)^2+l'(s)^2}ds.$$

Observe that the equality, $\text{area}(D(q,\alpha))=\text{area}(D_R(q,\alpha))$, defines $R$ implicitly as a function of $\alpha$ and implies

$$\int_0^\alpha\int_0^{l(s)}A(t,s)dtds = \int_0^\alpha\int_0^{R(\alpha)}A(t,s)dtds. \tag{6}$$

The objective is to study, for small $\alpha$, the sign of the function

$$\begin{aligned}F(\alpha) &:= \text{length}(\Gamma(q,\alpha))-\text{length}(S_R(\overline{q},\alpha))\\ &= \int_0^\alpha\sqrt{A(l(s),s)^2+l'(s)^2}ds-\int_0^\alpha A(R(\alpha),s)ds.\end{aligned}$$

Since $F(0)=0$, the sign of $F(\alpha)$, for $\alpha$ small, is the same as the sign of

$$F'(\alpha) = \sqrt{A(l(\alpha),\alpha)^2+l'(\alpha)^2}-A(R(\alpha),\alpha)-R'(\alpha)\int_0^\alpha\partial_t A(R(\alpha),s)ds,$$

and this is the same as that of

$$G(\alpha) := A(l(\alpha),\alpha)^2+l'(\alpha)^2-\left(A(R(\alpha),\alpha)+R'(\alpha)\int_0^\alpha\partial_t A(R(\alpha),s)ds\right)^2$$

(because $A(R(\alpha),\alpha)+R'(\alpha)\int_0^\alpha\partial_t A(R(\alpha),s)ds$ is positive for $\alpha$ small).

Theorem 1.1 is an immediate consequence of the following results:

**Theorem 2.1** *Let $M^2$ be a Riemannian manifold whose infinitesimal change of volume function with respect to $p$ is radial in $p\triangleleft\Gamma$ and suppose that $\Gamma$ is radially connected to $p$. Then there is $\delta>0$ such that for $0<\alpha\le\delta$, the equality,* $\text{area}(D(q,\alpha))=\text{area}(D_R(\overline{q},\alpha))$, *implies*

$$\text{length}(\Gamma(q,\alpha))\ge\text{length}(S_R(\overline{q},\alpha)).$$

*If the equality holds for every $0<\alpha\le\delta_1$, then $q=\overline{q}$ and $\Gamma(q,\alpha)=S_R(\overline{q},\alpha)$.*

*Proof of Theorem* 2.1 In this case the function $A(t,s)=A(t)$ depends only on $t$. Suppose that $l(s)$ has the Taylor expansion $l(s)=l_0+l_ks^k+l_{k+1}s^{k+1}+\cdots$ with $k>0$ and $l_k\ne 0$. It is also known that $l_0=l(0)\ne 0$. The function $\alpha\to\int_0^\alpha\int_0^{l(s)}A(t)dtds$ has the Taylor expansion

$$\left(\int_0^{l_0}A(t)dt\right)\alpha+l_kA(l_0)\frac{\alpha^{k+1}}{k+1}+l_{k+1}A(l_0)\frac{\alpha^{k+2}}{k+2}+\cdots+l_{2k-2}A(l_0)\frac{\alpha^{2k-1}}{2k-1}+\cdots.$$

If $R(\alpha)=R_0+R_1\alpha+R_2\alpha^2+\cdots$ then we have that the function

$$\alpha\to\int_0^\alpha\int_0^{R(\alpha)}A(t)dtds=\alpha\int_0^{R(\alpha)}A(t)dt$$

has the Taylor expansion

$$\left(\int_0^{R_0} A(t)dt\right)\alpha + R_1 A(R_0)\alpha^2 + R_2 A(R_0)\alpha^3 + \cdots + R_{2k-2}A(R_0)\alpha^{2k-1} + \cdots .$$

By (6), we get

$$R_0 = l_0, R_1 = 0, \ldots, R_{k-1} = 0, R_k = \frac{l_k}{k+1}, \ldots, R_{2k-2} = \frac{l_{2k-2}}{2k-1}, \ldots . \tag{7}$$

The function $G(\alpha)$ is, in this case,

$$A(l(\alpha))^2 + l'(\alpha)^2 - (A(R(\alpha)) + \alpha R'(\alpha)A'(R(\alpha)))^2.$$

By a straightforward computation we get the Taylor expansions

$$\begin{aligned} A(l(\alpha))^2 + l'(\alpha)^2 = {} & A(l_0)^2 + 2A(l_0)A'(l_0)l_k\alpha^k + 2A(l_0)A'(l_0)l_{k+1}\alpha^{k+1} + \cdots \\ & + 2A(l_0)A'(l_0)l_{2k-3}\alpha^{2k-3} \\ & + \left(2A(l_0)A'(l_0)l_{2k-2} + k^2{l_k}^2\right)\alpha^{2k-2} + \cdots \\ (A(R(\alpha)) + \alpha R'(\alpha)A'(R(\alpha)))^2 = {} & A(R_0)^2 + 2(k+1)A(R_0)A'(R_0)R_k\alpha^k + \cdots \\ & + 2(2k-2)A(R_0)A'(R_0)R_{2k-3}\alpha^{2k-3} \\ & + (2(2k-1)A(R_0)A'(R_0)R_{2k-2})\,\alpha^{2k-2} + \cdots . \end{aligned}$$

Then, using these results and the relations between $R_i$ and $l_i$ given in (7), we get $G(\alpha) = k^2 l_k^2\alpha^{2k-2} + \cdots$ and Theorem 2.1 follows.

Theorem 1.3 is an immediate consequence of the following results:

**Theorem 2.2** *Let $M^2$ be a riemannian manifold. Let $p, q \in M$, $q \notin \mathrm{cut}(p)$ and let $a \in S^1 \subset T_pM$ such that $q = \exp_p l_a a$. Suppose that $(A\partial_t\partial_s A - \partial_t A\partial_s A)(l_a, 0) \neq 0$, where $A(t,s) = t\theta(t, \varphi(s))$. Then there is a curve $\Gamma \subset M$ with $\delta > 0$ such that with $q \in \Omega$ and, for $0 < \alpha \leq \delta$, if $\mathrm{area}(D(q,\alpha)) = \mathrm{area}(D_R(\overline{q},\alpha))$, then $\mathrm{length}(\Gamma(q,\alpha)) < \mathrm{length}(S_R(\overline{q},\alpha))$.*

**Lemma 2.3** *The infinitesimal change of the volume function with respect to $p$ is radial if and only if*

$$(A\partial_t\partial_s A - (\partial_t A)(\partial_s A))(t,s) = 0, \quad \forall\, s \in [-\pi, \pi], \quad \forall\, t < f(s), \tag{8}$$

*where $f(s)$ is the first positive zero of the function $t \to A(t,s)$.*

*Proof of Theorem* 2.2 We can construct a curve $\Gamma \subset M$ with $q \in \Gamma$ which locally can be parametrized by $s \to \exp_p l(s)\varphi(s)$, with $l(s) = l_0 + l_k s^k + l_{k+1}s^{k+1} + \cdots$, $l_0 = l_a$ and $l_k \neq 0$.

Similarly to the proof of Theorem 2.1, by comparing the Taylor expansions of the functions in the equality (6), we obtain

$$\begin{aligned} R_0 &= l_0 \\ R_1 &= 0 \\ &\cdots \\ R_{k-1} &= 0 \\ R_k &= \frac{l_k}{k+1} \\ R_{k+1} &= \frac{l_{k+1}}{k+2} + \frac{l_k\partial_s A(l_0,0)}{A(l_0,0)}\left(\frac{1}{k+2} - \frac{1}{2(k+1)}\right). \end{aligned}$$

We take $k > 3$. Using the Taylor expansions of the functions $\alpha \to A(l(\alpha),\alpha)^2 + l'(\alpha)^2$ and $\alpha \to \left(A(R(\alpha),\alpha) + R'(\alpha)\int_0^\alpha \partial_t A(R(\alpha),s)ds\right)^2$ and the relations between $R_i$ and $l_i$ we deduce that

$$G(\alpha) = \left(\frac{k}{k+1}\right) l_k\,(A\partial_t\partial_s A - (\partial_t A)(\partial_s A))\,(l_0,0)\alpha^{k+1} + \cdots .$$

By hypothesis $(A\partial_t\partial_s A - (\partial_t A)(\partial_s A))\,(l_0,0) \neq 0$, so it is possible to choose $l_k$ such that the first non-zero term of the Taylor expansion of $G(\alpha)$ is negative. Then the curve $\Gamma$ constructed in this form satisfies the conclusion of Theorem 2.2.

**Remark** Observe that $l'(\alpha)^2 = k^2 {l_k}^2 \alpha^{2k-2} + \cdots$. If $k > 3$, $l'(\alpha)$ does not contribute to the first non-zero term of the Taylor expansion of $G(\alpha)$. If $k = 1$ or $k = 2$ then $G(\alpha) = l_1^2 + \cdots$ and $G(\alpha) = 4l_2^2\alpha^2 + \cdots$ and, in these cases, it is not possible to construct a curve $\Gamma$ as above satisfying the thesis of Theorem 2.2. In fact, it is easily seen that these conditions are equivalent to the conditions (a) and (b) respectively of Theorem 4.1. If $k = 3$ then

$$G(\alpha) = \left(\frac{3}{4} l_3 \left(A\partial_t\partial_s A - (\partial_t A)(\partial_s A)\right)(l_0, 0) + 9{l_3}^2\right)\alpha^4 + \cdots .$$

Choosing $l_3$ sufficiently near to 0 and with the sign opposite to

$$\left(A\partial_t\partial_s A - (\partial_t A)(\partial_s A)\right)(l_0, 0)$$

a counterexample can also be constructed.

*Proof of Lemma* 2.3 First, let us consider the following formulae

$$\partial_s\partial_t \log(A) = \partial_s\left(\frac{\partial_t A}{A}\right) = \frac{A\partial_s\partial_t A - \partial_t A \partial_s A}{A^2}, \tag{9}$$

$$\partial_t \log(A) = \partial_t \log(t\theta) = \partial_t \log(t) + \partial_t \log(\theta) = \frac{1}{t} + \partial_t \log(\theta), \tag{10}$$

$$\log(\theta(s,t)) - \log(\theta(s,0)) = \int_0^t \partial_t \log(\theta(s,t))dt, \tag{11}$$

$$\theta(s,0) = 1. \tag{12}$$

From (9) $A\partial_s\partial_t A - \partial_t A\partial_s A = 0$ if and only if $\partial_t \log(A)$ not depending on $s$, which, from (10), is equivalent to $\partial_t \log(\theta)$ does not depend on $s$, and from (11) and (12), $\partial_t \log(\theta)$ does not depend on $s$ if and only if $\theta(s,t)$ does not depend on $s$, which finishes the proof of the lemma.

## 3 Formulae for the Volume of a Cone and Its Base

In order to prove Theorem 1.6 we extend the formulae for the areas and lengths of sectors in §2 to the volumes of $C\Omega(q,r)$ and $\Omega(q,r)$. Let $\theta(t,u)$ be the infinitesimal change of volume function with respect to $p$ which is defined by

$$\omega(\exp_p tu)) = t^{n-1}\theta(t,u)dtdu,$$

where $\omega$ and $du$ are the Riemannian volume elements of $M$ and $S^{n-1} \subset T_pM$.

From the definitions of $C\Omega(q,r)$ and $\theta(t,u)$, it is easy to see (cf. [6], for instance) that

$$\text{volume}(C\Omega(q,r)) \quad = \quad \int_{B_r(a)} \int_0^{l(u)} t^{n-1}\theta(t,u)dtdu.$$

From the definition of $\Omega(q,r)$, following standard arguments (see, for instance [11] or [12] (5.2)) it easy to see that

$$\text{volume}(\Omega(q,r)) \quad = \quad \int_{B_r(a)} \frac{l(u)^{n-1}\theta(l(u),u)}{|\langle {\gamma_u}'(l(u)), \nu_u\rangle|}dtdu,$$

where $\gamma_u(t) = \exp_p tu$ for every $u \in B_r(a)$ and $\nu_u$ is the unit vector normal to $\Omega(q,r)$ at $\exp_p l(u)u$.

To get more useful formulae for these volumes we introduce, for $n \geq 2$, the unit sphere $S^{n-1}$ in $T_pM$, $a \in S^{n-1}$, the unit sphere $S^{n-2}$ in $T_aS^{n-1}$ and the function $\varphi\colon ]0,\pi[\times S^{n-2} \to S^{n-1}$ defined by $\varphi(s,v) := \exp_a^S(sv)$, where $\exp^S$ is the exponential map in $S^{n-1}$. Writing $\theta(t,s,v) := \theta(t,\varphi(s,v))$, $l(s,v) := l(\varphi(s,v))$ and using the function $\varphi$ to compute the integral along $B_r(a)$ and the well-known expression of the infinitesimal change of volume function in $S^{n-1}$, we get

$$\text{volume}(C\Omega(q,r)) = \int_{S^{n-2}} \int_0^r \int_0^{l(s,v)} (\sin s)^{n-2} t^{n-1}\theta(t,s,v)dtdsdv,$$

$$\text{volume}(\Omega(q,r)) = \int_{S^{n-2}} \int_0^r \frac{(\sin s)^{n-2} l(s,v)^{n-1}\theta(l(s,v),s,v)}{|\langle {\gamma_{\varphi(s,v)}}'(l(s,v)), \nu_{\varphi(s,v)}\rangle|}dsdv. \tag{13}$$

When $\Omega = S_R(p)$ we have

$$\mathrm{vol}(CS_R(\overline{q},r)) = \int_{S^{n-2}}\int_0^r\int_0^R (\sin s)^{n-2}t^{n-1}\theta(t,s,v)dtdsdv,$$
$$\mathrm{vol}(S_R(\overline{q},r)) = R^{n-1}\int_{S^{n-2}}\int_0^r (\sin s)^{n-2}\theta(R,s,v)dsdv. \tag{14}$$

## 4 Proof of Theorem 1.6

Theorem 1.6 is an immediate consequence of the following result:

**Theorem 4.1** *Suppose that the function $u \to l(u)$ is $C^\infty$ in a neighborhood of $a$ and that*

(a) $|\langle {\gamma_a}'(l_a), \nu_a\rangle| < 1$ *or*

(b) $|\langle {\gamma_a}'(l_a), \nu_a\rangle| = 1$ *and* $\int_{S^{n-2}} \frac{d}{ds}|\langle {\gamma_{\varphi(s,v)}}'(l(s,v)), \nu_{\varphi(s,v)}\rangle|(0,v)dv < 0$.

*Then there exists $\delta > 0$ such that for $0 < r \le \delta$, if* $\mathrm{vol}(C\Omega(q,r)) = \mathrm{vol}(CS_R(\overline{q},r))$, *then*

$$\mathrm{vol}(\Omega(q,r)) > \mathrm{vol}(S_R(\overline{q},r)).$$

Observe that $\delta$ depends on $p$, $q$ and $\Omega$, and that $\Omega$ is compact, $\delta$ depends only on $p$ and $\Omega$.

Moreover the equality

$$\mathrm{vol}(C\Omega(q,r)) \quad = \quad \mathrm{vol}(CS_R(\overline{q},r)) \tag{15}$$

defines implicitly $R$ as a function of $r$.

Theorem 4.1 is not true if conditions (a) and (b) fail (see the remark at the end of the proof of Theorem 2.2). For $\alpha \ll 2\pi$ under the hypothesis (a) or (b), this theorem partially extends Choe–Gulliver's results ([7, 8]). The results of Bray [9] and Morgan–Ritoré [10] have nonempty intersection with Theorem 4.1.

Theorem 4.1 is a result for small $r$. For big $r$ (near to $\pi$) we have the following:

Let $\mathbb{K}^n(\lambda)$ be the simply connected manifold of dimension $n$ and constant sectional curvature $\lambda \in \mathbb{R}$. As the solution of the general isoperimetric problem in $\mathbb{K}^n(\lambda)$ is the geodesic ball then, by continuity, if $Q$ is a connected domain in $M$ with smooth boundary $\Omega$ which is not a geodesic ball, $p \in M$ is an interior point of $Q$, $\Omega(q,\pi) = \Omega$ for some $q \in \Omega$ and $C\Omega(q,\pi) = Q$, then there exists $\delta > 0$ such that for $0 < r < \delta$, if $\mathrm{vol}(C\Omega(q,\pi - r)) = \mathrm{vol}(CS_R(\overline{q},\pi - r))$, then

$$\mathrm{vol}(\Omega(q,\pi - r)) > \mathrm{vol}(S_R(\overline{q},\pi - r)).$$

Moreover, if $p \in M$ is an interior point of $Q$ such that $d(p,\Omega) = \mathrm{diameter}(Q)/2$ and the condition that $Q$ is not a geodesic ball is eliminated, then the strict inequality above must now be an inequality and the equality implies that $Q$ is a geodesic ball and $\Omega$ is its boundary.

From now on $\partial_i$, $i = 1, 2$, will mean the derivative with respect to the $i$-th argument.

In order to prove Theorem 4.1, we calculate the Taylor expansion of the implicit function $R(r)$ defined by (15).

**Proposition 4.2** *We have the Taylor expansion*

$$R(r) = l_a + \left(\frac{n-1}{n}\frac{\int_{S^{n-2}} l'(0,v)dv}{\mathrm{vol}S^{n-2}}\right)r + \cdots,$$

*where $'$ denotes the derivative with respect to $s$.*

*Proof* By a straightforward computation, the Taylor expressions of $\frac{d}{dr}\mathrm{vol}(C\Omega(q,r))$ and $\frac{d}{dr}\mathrm{vol}(CS_R(\overline{q},r))$ are equal (by (15)), then, by comparing the terms in $r^{n-2}$,

$$\mathrm{vol}(S^{n-2})\Big(\int_0^{l_a} t^{n-1}\theta(t,a)dt\Big)r^{n-2} = \mathrm{vol}(S^{n-2})\Big(\int_0^{R(0)} t^{n-1}\theta(t,a)dt\Big)$$

and thus $R(0) = l_a$.

By comparing the terms in $r^{n-1}$

$$\begin{aligned}
&{l_a}^{n-1}\theta(l_a,a)\int_{S^{n-2}} l'(0,v)dv + \int_{S^{n-2}}\int_0^{l_a} t^{n-1}\partial_2\theta(t,0,v)dtdv \\
&\quad = R(0)^{n-1}R'(0)\theta(R(0),a)\mathrm{vol}(S^{n-2}) + \int_{S^{n-2}}\int_0^{R(0)} t^{n-1}\partial_2\theta(t,0,v)dtdv \\
&\quad\quad + \frac{1}{n-1}R(0)^{n-1}R'(0)\theta(R(0),a)\mathrm{vol}(S^{n-2}).
\end{aligned}$$

As $R(0) = l_a$ we get

$${l_a}^{n-1}\theta(l_a,a)\int_{S^{n-2}} l'(0,v)dv = \frac{n}{n-1}{l_a}^{n-1}R'(0)\ \theta(l_a,a)\ \mathrm{vol}(S^{n-2})$$

and thus

$$R'(0) = \left(\frac{n-1}{n}\right)\frac{\int_{S^{n-2}} l'(0,v)dv}{\mathrm{vol}(S^{n-2})}.$$

The statement of the proposition follows.

*Proof of Theorem* 4.1 We will find the Taylor expansion of the function $F(r)$ defined by

$$F(r) = \mathrm{vol}(\Omega(q,r)) - \mathrm{vol}(S_R(\overline{q},r)).$$

First we give the Taylor expansion of (see (13))

$$\mathrm{volume}(\Omega(q,r)) = \int_{S^{n-2}}\int_0^r (\sin s)^{n-2} l(s,v)^{n-1}\theta(l(s,v),s,v)g(s,v)dsdv,$$

where $g(s,v) := \left(|\langle({\gamma_{\varphi(s,v)}}'(l(s,v)),\nu_{\varphi(s,v)}\rangle|\right)^{-1}$.

As

$$\begin{aligned}
(\sin s)^{n-2} &= s^{n-2} - \frac{n-2}{6}s^n + \cdots \\
l(s,v)^{n-1} &= {l_a}^{n-1} + (n-1){l_a}^{n-2}l'(0,v)s + \cdots \\
\theta(l(s,v),s,v) &= \theta(l_a,a) + (\partial_1\theta(l_a,0,v)l'(0,v) + \partial_2\theta(l_a,0,v))\,s + \cdots \\
g(s,v) &= g(0,v) + g'(0,v)s + \cdots \\
&= \frac{1}{|\langle{\gamma_a}'(l_a),\nu_a\rangle|} + g'(0,v)s + \cdots
\end{aligned}$$

then

$$\begin{aligned}
\mathrm{volume}(\Omega(q,r)) &= \int_{S^{n-2}}\int_0^r \Bigg\{\frac{{l_a}^{n-1}\theta(l_a,a)}{|\langle{\gamma_a}'(l_a),\nu_a\rangle|}s^{n-2} + \Bigg[\frac{(n-1){l_a}^{n-2}l'(0,v)\theta(l_a,a)}{|\langle{\gamma_a}'(l_a),\nu_a\rangle|} \\
&\quad + {l_a}^{n-1}\theta(l_a,a)g'(0,v) \\
&\quad + \frac{{l_a}^{n-1}(\partial_1\theta(l_a,0,v)l'(0,v) + \partial_2\theta(l_a,0,v))}{|\langle{\gamma_a}'(l_a),\nu_a\rangle|}\Bigg]s^{n-1} + \cdots\Bigg\}dsdv \\
&= \left[\frac{\mathrm{vol}(S^{n-2}){l_a}^{n-1}\theta(l_a,a)}{(n-1)|\langle{\gamma_a}'(l_a),\nu_a\rangle|}\right]r^{n-1} + \Bigg[\frac{(n-1){l_a}^{n-2}\theta(l_a,a)}{n|\langle{\gamma_a}'(l_a),\nu_a\rangle|}\int_{S^{n-2}} l'(0,v)dv \\
&\quad + \frac{{l_a}^{n-1}\theta(l_a,a)}{n}\int_{S^{n-2}} g'(0,v)dv\frac{{l_a}^{n-1}\partial_1\theta(l_a,a)}{n|\langle{\gamma_a}'(l_a),\nu_a\rangle|}\int_{S^{n-2}} l'(0,v)dv \\
&\quad + \frac{{l_a}^{n-1}}{n|\langle{\gamma_a}'(l_a),\nu_a\rangle|}\int_{S^{n-2}}\partial_2\theta(l_a,0,v)dv\Bigg]r^n + \cdots.
\end{aligned}$$

Similarly, the Taylor expansion of

$$\mathrm{vol}(S_{R(r)}(\overline{q},r)) = R(r)^{n-1}\int_{S^{n-2}}\int_0^r (\sin s)^{n-2}\theta(R(r),s,v)\ ds\ dv$$

(see (14)) is

$$\mathrm{vol}(S_{R(r)}(\overline{q},r)) = \left[\frac{1}{n-1}\mathrm{vol}(S^{n-2})R(0)^{n-1}\theta(R(0),a)\right]r^{n-1} + [R(0)^{n-2}R'(0)\theta(R(0),a)\mathrm{vol}(S^{n-2}) + \frac{1}{n}R(0)^{n-1}\int_{S^{n-2}}\partial_2\theta(R(0),0,v)dv + \frac{1}{n-1}R(0)^{n-1}R'(0)\int_{S^{n-2}}\partial_1\theta(R(0),0,v)dv]r^n + \cdots .$$

By Proposition 4.2

$$\mathrm{vol}(S_{R(r)}(\overline{q},r)) = \left[\frac{\mathrm{vol}(S^{n-2}){l_a}^{n-1}\theta(l_a,a)}{n-1}\right]r^{n-1} + \left[\frac{n-1}{n}{l_a}^{n-2}\theta(l_a,a)\int_{S^{n-2}}l'(0,v)dv + \frac{{l_a}^{n-1}}{n}\int_{S^{n-2}}\partial_2\theta(l_a,0,v)dv + \frac{1}{n}{l_a}^{n-1}\partial_1\theta(l_a,a)\int_{S^{n-2}}l'(0,v)dv\right]r^n + \cdots .$$

Suppose that $|\langle {\gamma_a}'(l_a),\nu_a\rangle| < 1$. Then the first non-zero term of the Taylor expansion of $F(r)$ is

$$\frac{\mathrm{vol}(S^{n-2}){l_a}^{n-1}\theta(l_a,a)}{n-1}\left[\frac{1}{|\langle {\gamma_a}'(l_a),\nu_a\rangle|}-1\right]r^{n-1}.$$

As

$$\frac{\mathrm{vol}(S^{n-2}){l_a}^{n-1}\theta(l_a,a)}{n-1}\left[\frac{1}{|\langle {\gamma_a}'(l_a),\nu_a\rangle|}-1\right] > 0,$$

then there exists $\delta > 0$ such that for $0 < r \leq \delta$, $F(r) \geq 0$. In fact, it is possible to choose $\delta$ such that for $0 < r < \delta$, $F(r) > 0$.

If $|\langle {\gamma_a}'(l_a),\nu_a\rangle| = 1$ and $\int_{S^{n-2}} b(v)dv < 0$ with $b(v) = \frac{d}{ds}|\langle {\gamma_{\varphi(s,v)}}'(l(\varphi(s,v)),\nu_{\varphi(s,v)}\rangle|(0,v)$, then $|\langle {\gamma_{\varphi(s,v)}}'(l(\varphi(s,v)),\nu_{\varphi(s,v)}\rangle|$ has Taylor expansion $1 + b(v)s + \cdots$.

Then

$$g(s,v) = \left(|\langle {\gamma_{\varphi(s,v)}}'(l(s,v)),\nu_{\varphi(s,v)}\rangle|\right)^{-1} = 1 - b(v)s + \cdots$$

and the first non-zero term of the Taylor expansion of $F(r)$ is

$$\left[\frac{{l_a}^{n-1}\theta(l_a,a)}{n}\int_{S^{n-2}}g'(0,v)dv\right]r^n = -\left[\frac{{l_a}^{n-1}\theta(l_a,a)}{n}\int_{S^{n-2}}b(v)dv\right]r^n$$

which is strictly positive for $r > 0$. Then there exists $\delta > 0$ such that for $0 < r \leq \delta$, $F(r) \geq 0$. In fact, it is possible to choose $\delta$ such that for $0 < r < \delta$, $F(r) > 0$.

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