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Fractal axicons

Juan A. Monsoriu [a,*], Carlos J. Zapata-Rodríguez [b], Walter D. Furlan [b]

[a] Departamento de Física Aplicada, Universidad Politécnica de Valencia, Camino de Vera sn, E-46022 Valencia, Spain
[b] Departamento de Óptica, Universidad de Valencia, E-46100 Burjasstot (Valencia), Spain

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Abstract

Cantor rings are rotational symmetric pupils that are generated from a Cantor set of a given level of growth. These pupils have certain fractal properties. For example, it is known that when illuminated by a general spherical wavefront they provide self-similar patterns at transverse planes in the Fraunhofer region. In this paper, we study the response of Cantor rings when illuminated by a Bessel light beam conforming what we call fractal axicons. It is shown that, with this kind of illumination a close replica of the radial profile of the pupil is obtained along the optical axis, i.e., we show that the axial behaviour of these pupils has self-similarity properties that can be correlated to those of the diffracting aperture. The influence of several construction parameters is numerically investigated.

Keywords: Fractals; Cantor set; Bessel beams; Axial irradiance; Axicons

1. Introduction

Certain natural phenomena exhibit distinctive features that can be associated with the concept of fractal. Fractals are self-similar structures that are invariant under a change of scale which, in addition, have a fractional dimension. Fractals have been known over the past centuries, but certainly it was not until the development of computer science that they became a matter of great interest for scientists in many fields [1].

In optics, the diffraction properties of fractal objects have been studied extensively ranging from simple one-dimensional objects to complex 2D systems (See for example Refs. [2,3] and the references therein), including fractal zone plates [4,5], which allow the fractal focusing of light among other properties. A special case of 2D rotationally symmetric self-similar objects are the Cantor rings (CRs), or Cantor ring diffractals as coined by Jaggard and Jaggard [6]. In their original paper these authors found that the transverse diffraction patterns produced in the far field by CRs exhibit scaling features that are typical of regular fractal structures. Lately the axial behaviour of CRs has also been investigated, but in this case, it has been shown that the irradiance produced by a CR does not present any fractal structure of the aperture [7].

There are still many other interesting features of the CRs that have not been developed but could be significant for future applications. Of particular interest are the diffraction properties given by these fractal structures when illuminated by Bessel light beams. These beams produce a uniform axial distribution with an extremely narrow central peak in the transverse direction [8]. This feature offers, for example, enhanced optical guiding possibilities, which have important restrictions for other types of illumination (in particular, for a Gaussian beam the physical limit is the Rayleigh range). The use of Bessel beams is then optimal for optical trapping and manipulation of microscopic particles and biological cells [9], for optical coherence tomography [10] and more generally for experiments in nonlinear optics [11]. Experimental proposals to obtain nondiffracting Bessel beams are numerous in the literature [12,13], among which we find very reliable the holographic elements [14]

* Corresponding author. Tel.: +34 963877525; fax: +34 963877189.
E-mail address: jmonsori@fis.upv.es (J.A. Monsoriu).

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and spatial light modulators [15]. The proper combination of these elements with a Cantor rings spatial filter constitutes what we call a fractal axicon, that is, an optical arrangement that focuses light onto the axis with fractal properties.

In this work the axial behaviour of a Bessel beam impinging on a CR is studied. In Section 2, we use the Fresnel diffraction integral to evaluate the axial intensity distribution of an apertured azimuthally-symmetric Bessel beam. We demonstrate that, from the irradiance distribution of the pupil aperture, a nonexact but close replica of it is obtained along the optical axis. Also we give the practical limitations to observe this exclusive behaviour. In Section 3, we present the procedure we followed for the synthesis of the CRs. In Section 4, we perform the numerical analysis of CRs illuminated by a Bessel beam. The influence of the effective Fresnel number [16] associated to a CR on the axial irradiance is investigated. Additionally, the self-similarity of the axial irradiance is compared with the self-similarity of CR itself. Finally, in Section 5 the main results of this paper are outlined and some applications are proposed.

2. Axial intensity distribution of apertured Bessel beams

Bessel beams are solutions of the scalar Helmholtz equation expressed in cylindrical coordinates [7]. They are interpreted as a linear combination of propagating plane waves whose wave vectors $\boldsymbol{k}$ have the same projection $\beta = k\cos\theta$ onto the axis of propagation. In particular for a wave function with azimuthal symmetry, the solely nonsingular solution of the wave equation is given by

$$U_0(r,z) = \exp(jkz\cos\theta)J_0(kr\sin\theta), \tag{1}$$

where J_0 is the Bessel function of first kind and zero-order, and the amplitude on the axis is assumed to be the unity. From Eq. (1), we observe that the wavefield presents an invariant transverse pattern (except for a constant phase) despite of propagation. The beam amplitude at the optical axis is also invariant.

When limited by an aperture, zero-order Bessel beams are no longer free-space diffraction modes and consequently the common diffraction spreading is observed beyond a certain distance. However, in the near field, the transverse amplitude distribution closely resembles a zero-order Bessel function. Additionally, the irradiance along the optical axis is almost unaltered along a certain distance, producing a focal line [17]. From the experimental point of view, the family of optical arrangements which perform an on-axis linear focus are known as axicons [18,19]. Interestingly, it has been stated that the diffraction characteristics of the focal region in refractive axicons are determined by the so-called Fresnel number such as, for example, the ability of generating a nondiffracting Bessel beam and the focal shift of the irradiance maximum, with the consequent reduction of the focal length named focal squeeze [16].

According to the Fresnel–Kirchhoff diffraction integral, the on-axis irradiance distribution of a rotationally symmetric Bessel beam limited by a circular aperture of radius a is given by

$$I(z) = \left(\frac{k}{z}\right)^2 \left| \int_0^a J_0(kr_0\sin\theta)\exp\left(\mathrm{j}\frac{k}{2z}r_0^2\right) r_0 \mathrm{d}r_0 \right|^2, \tag{2}$$

where z is the axial distance from the aperture to the observation point. Note that under the substitution $\sin\theta = r/z$, the integral in Eq. (2) is analogous to the Fresnel diffraction pattern of a circular aperture illuminated with a uniform plane wave, for which the geometrical approximation in the near field predicts a uniform irradiance distribution for radial values $r \leqslant a$ and zero for $r > a$ [20]. On the analogy to this situation, we can state that the axial irradiance generated by a Bessel beam limited by a circular aperture is constant within the interval

$$z \leqslant z_{\max} = \frac{a}{\sin\theta}, \tag{3}$$

and is zero for values beyond $z_{\max}$. Obviously, this prediction holds when $ka\sin\theta$ tends to infinity, but it is still accurate if $ka\sin\theta$ is several orders of magnitude higher than unity. Following a previous work [16], we may define the Fresnel number for apertured Bessel beams as

$$N = a\sin\theta/\lambda. \tag{4}$$

Accordingly we should impose that N is much higher than the unity.

Let us now consider the irradiance at a given point on the optical axis provided by a rotationally-invariant purely-absorbing pupil with an amplitude transmittance $p(r)$, which is illuminated by a monochromatic nonuniform beam of amplitude $U_0(r,0)$ (see Fig. 1). Within the Fresnel approximation we find that

$$I(z) = \left(\frac{k}{z}\right)^2 \left| \int_0^\infty U_0(r_0,0)p(r_0)\exp\left(\mathrm{j}\frac{k}{2z}r_0^2\right) r_0 \mathrm{d}r_0 \right|^2. \tag{5}$$

Again, we may use the geometrical approximation under the assumption of very high Fresnel numbers, and we

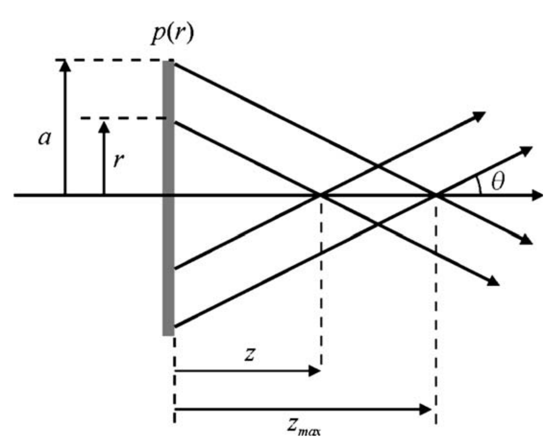

Fig. 1. On-axis irradiance produced by a Bessel beam, of propagating angle θ, impinging on a pupil aperture of transmittance $p(r)$.

expect that $I(z) \approx |p(z\sin\theta)|^2$. In the case of moderately high Fresnel numbers, we should also impose that the transmittance function $p(r)$ varies slowly. Alternatively, we may introduce an effective Fresnel number in order to consider the beam-shaping effect of $p(r)$ in the diffracted field [21]. In particular, an annular pupil of width $\delta = a_{\text{out}} - a_{\text{in}}$ is accurately characterized by an effective Fresnel number given by

$$N_{\text{e}} = \delta \sin\theta/\lambda. \tag{6}$$

Also note that in the paraxial regime we may use the approximations $\sin\theta \approx \tan\theta \approx \theta$. Then we obtain a non-perfect but close replica of the transverse intensity distribution along the optical axis, with a magnitude given by the inverse of θ. Every symmetry property that exhibits the pupil along the radial direction is approximately replicated on the axial intensity distribution as far as the effective Fresnel number is much higher than unity (ideally tending to infinity).

The above-mentioned general property that to the knowledge of the authors is original, represents one of the main results of this paper. In fact, in the geometrical approximation a Bessel beam can be seen as conical ray fans and, as seen in Fig. 1, the radial pupil features are then transferred to the optical axis. Additionally we exploit this feature to create fractal patterns along the optical axis. Next we focus our attention to the functions $p(r)$, constructed from different levels of a Cantor set.

3. Cantor rings

A CR is a 2D rotationally symmetric pupil that is generated from a 1D Cantor set to a level of growth. As an example, let us consider the construction of the regular triadic Cantor set shown in Fig. 2(a). The first step in the construction procedure consists in defining a straight-line segment of length a called initiator (stage $S = 0$). Next, at stage $S = 1$, the generator of the set is constructed by dividing the segment in three equal parts of length $a/3$ and removing the central one, in this case the scale factor is 1/3. Following this procedure in subsequent stages $S = 2, 3, \ldots$ is easy to see that, in general, at the stage S,

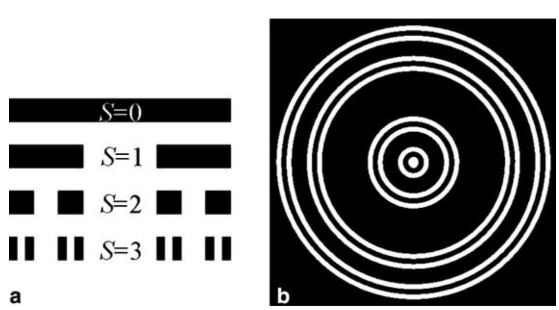

Fig. 2. (a) Triadic Cantor set for the levels $S = 0, 1, 2$, and 3. The structures for $S = 0$ and $S = 1$ are initiator and the generator of the set respectively. The bars represent the values of the 1D binary function $p(r)$. (b) CR generated from stage $S = 3$.

there are 2^S segments of length $a/3^S$ with $2^S - 1$ gaps in between. In Fig. 2(a), only the three first stages are shown for clarity (a similar procedure should be followed for Cantor sets other than triadic).

A CR is then a circularly symmetric pupil that is generated by rotating the Cantor bars distribution (at a given stage) around one of its extremes. Thus, a CR is characterized by a Cantor set distribution of transparent rings, which in our case, are of width $\delta = a/3^S$. In accordance with Eq. (6) the effective Fresnel number is then given by the expression $N_{\text{e}} = a\sin\theta/(3^S\lambda)$. In mathematical terms, a CR developed up to a certain growing stage S can be represented by a binary function transmittance $p(r)$ given by,

$$p(r) = \prod_{i=0}^{S} f\left(\frac{r}{a};\ q_{\text{i}}\right), \tag{7}$$

where a is the maximum extent of the pupil function and $f(x, q_i)$ is a Ronchi-type periodic binary function with period $q_{\text{i}} = 2/3^{\text{i}}$ that can be written as

$$f(x;\ q_{\text{i}}) = \text{rect}\left(x - \frac{1}{2}\right)\text{rect}\left[\text{mod}\left(x + \frac{q_{\text{i}} - 2}{2},\ q_{\text{i}}\right)\Big/ q_{\text{i}}\right], \tag{8}$$

where $\text{rect}(x) = 1$ for $|x| < 1/2$ and 0 otherwise. In this equation $\text{mod}(u, v)$ gives the remainder on division of u by v. Note that a CR can be understood as a circular Ronchi-type grating with period $q_{\text{i}}a$, but with some missing clear zones. Fig. 2(b) shows a CR generated from a triadic Cantor-set up to $S = 3$.

4. Results

The diffracted field along the optical axis in the region of interest was computed using Eq. (5), where $U_0(r_0, 0) = J_0(\frac{2\pi N_{\text{e}}}{\delta} r_0)$ and $p(r_0)$ are given in Eqs. (1) and (7), respectively, for the CRs with the values $S = 1, 2$, and 3 and two different pupil radius. The results for $a = 1$ cm and $a = 3$ cm are shown in Figs. 3 and 4. The first noticeable feature in these figures is that the axial irradiances reflect the fractal structure of the pupil along the radial coordinate. Additionally, as happens with the corresponding CRs, the axial irradiance for $S = 2$ is a scaled and replicated version of the axial irradiance for $S = 1$, and the same occurs for $S = 3$ respect to $S = 2$. The influence of the Fresnel number on the fractality of the axial irradiance exhibited by the CRs when illuminated by a Bessel beam can be observed from the comparison between Figs. 3 and 4 for the same value of S. Clearly, as stated in Section 2, as far as the effective Fresnel number increases, the resemblance between the radial profile of a given CR and the irradiance along the optical axis also increases. Obviously for the solution there is a compromise between high Fresnel numbers and the paraxial approximation, and for extremely high values of the Fresnel number we should use a vector formulation of the diffraction theory.

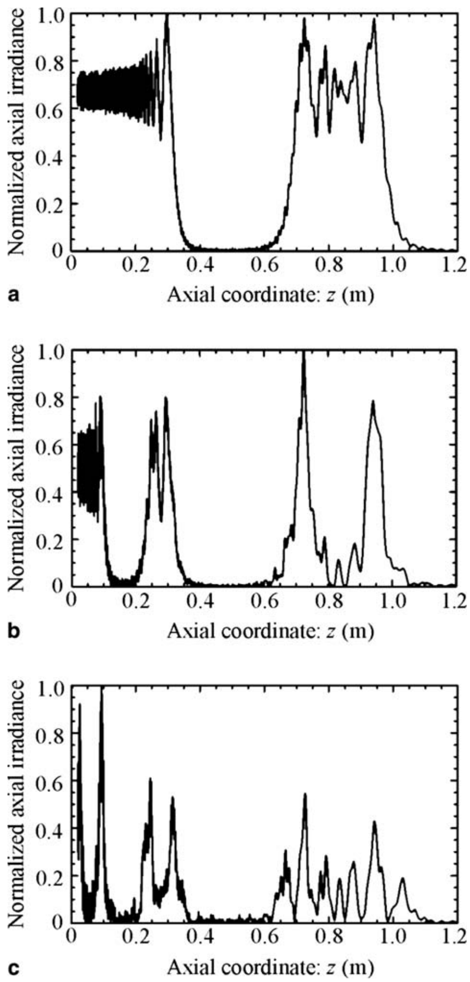

Fig. 3. Normalized axial irradiances given by CRs constructed with $a = 1$ cm and illuminated with a beam of wavelength $\lambda = 632.8$ nm. The following construction parameters were considered (a) $S = 1$, $\delta = 0.333$ cm, $N_e = 52.68$; (b) $S = 2$, $\delta = 0.111$ cm, $N_e = 17.56$, and (c) $S = 3$, $\delta = 0.037$ cm, $N_e = 5.85$. In all cases $z_{max} = 1$ m.

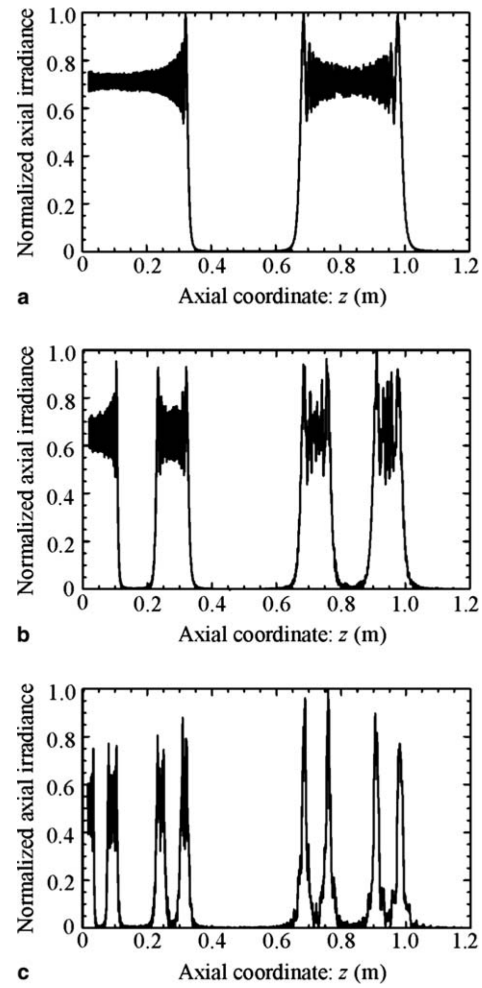

Fig. 4. Normalized axial irradiances given by CRs constructed with $a = 3$ cm and illuminated with a beam of wavelength $\lambda = 632.8$ nm. The following construction parameters were considered (a) $S = 1$, $\delta = 1$ cm, $N_e = 474.01$; (b) $S = 2$, $\delta = 0.333$ cm, $N_e = 158.00$, and (c) $S = 3$, $\delta = 0.111$ cm, $N_e = 52.67$. In all cases $z_{max} = 1$ m.

To investigate quantitatively the degree of self-similarity of the axial irradiances provided by a given CR with different Bessel beams, we use a correlation coefficient between the axial irradiance and its scaled version [2], i.e.,

$$C(\gamma) = \frac{\int_0^\infty I(z)I(z/\gamma)\mathrm{d}z}{\sqrt{\int_0^\infty I^2(z)dz \int_0^\infty I^2(z/\gamma)\mathrm{d}z}} \quad (9)$$

This function was computed for the irradiances of Figs. 3(c) and 4(c). The integrations were performed only over the corresponding axial region of interest. When $I(z)$ satisfies the strict axial self-similar property $I(z) = I(z/\gamma)$ the correlation coefficient (or simply, the self-similarity) is $C(\gamma) = 1$, what happens when $N \to \infty$. Correspondingly, lower degrees of self-similarity give values of $C(\gamma)$ lower than unity. Thus, if $C(\gamma)$ is plotted against the logarithm of the scale, some local maxima of the curve are expected to appear at $\gamma = 3^n$. The results are shown in Fig. 5(a) and (b). In these figures local maxima appear at $\gamma = 1, 3, 9,$ and 27. The smoother behaviour $C(\gamma)$ in Fig. 5(a) as compared with Fig. 5(b) is due to the low degree of correlation observed in Fig. 3(c) as compared with Fig. 4(c). For comparison, in Fig. 5(c) a similar correlation function was computed for the radial profile of the CR instead of the axial irradiance. It can be observed that the self similarity of the pupil function (Fig. 5(c)) and the self-similarity of the axial irradiance (Fig. 5(b)) have nearly the same profile.

5. Conclusions

The axial irradiance provided by CRs illuminated by Bessel beams has been examined. It was found that, contrary to their behaviour when illuminated by a spherical wavefront (which does not present self-similar properties), the axial irradiance produced by Bessel illumination reproduces the fractality of the pupil. This is much evident as the effective Fresnel number of the CR increases. This particular combination of CRs and Bessel beams conform a special kind of axicons that give fractal on-axis intensity distributions. Even for low values of the Cantor set S,

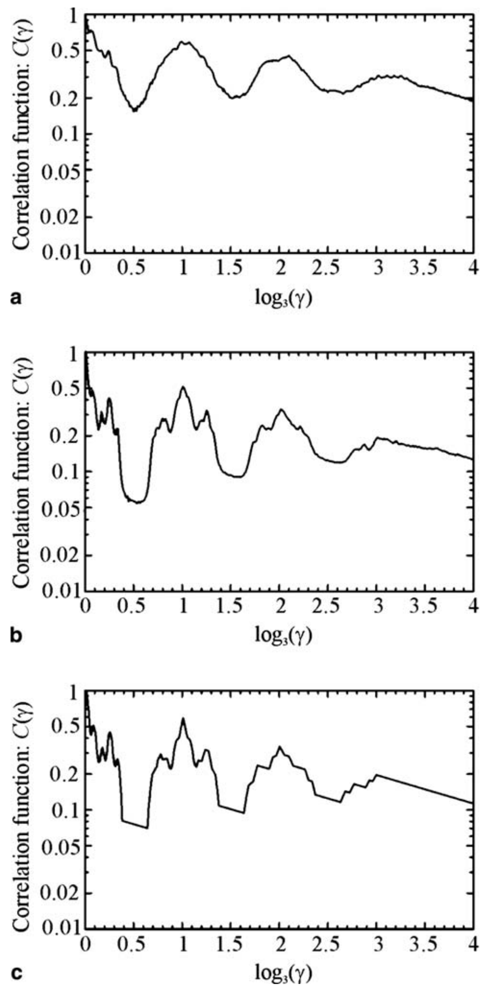

Fig. 5. Correlation function $C(\gamma)$ for: (a) the axial irradiance shown in Fig. 3(c); (b) the axial irradiance shown in Fig. 4(c), and (c) the binary function $p(r)$ of a CRs at stage $S = 3$.

the stepped profiles of the axial irradiance may be useful in several applications such optical trapping and optical metrology.

A common way of producing Bessel beams is by means of conical lenses (see Ref. [16] and references therein). This refractive element produces a linearly increasing rather than constant axial intensity. The use of these lenses together with a CR described in Section 3 generates a set of geometrical shadows along the optical axis with fractal distribution. Also, the intensity height of the light segments increases with the spatial axial coordinate producing a decrease of the correlation coefficient, $C(\gamma)$. However, the local maxima of this function appear at the same scaling factors. Thus, the self-similarity is significant, though is reduced in comparison with the studied case.

As mentioned in the introduction, a monochromatic plane wave can be transformed into a Bessel beam by various optical elements. When light pulses are considered instead of monochromatic radiation, the field is a spectral superposition of Bessel beams whose cone angle depends on the wavelength and consequently also changes the propagation depth. The study of interaction of such polychromatic beams with the Cantor rings is one of our future goals.

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References

[1] B.B. Mandelbrot, The Fractal Geometry of Nature, Freeman, San Francisco, CA, 1982.
[2] Y. Sakurada, J. Uozumi, T. Asakura, Pure Appl. Opt. 1 (1992) 29.
[3] O. Trabocchi, S. Granieri, W.D. Furlan, J. Mod. Opt. 48 (2001) 1247.
[4] G. Saavedra, W.D. Furlan, J.A. Monsoriu, Opt. Lett. 28 (2003) 971.
[5] J.A. Monsoriu, G. Saavedra, W.D. Furlan, Opt. Express 12 (2004) 4227.
[6] A.D. Jaggard, D.L. Jaggard, Opt. Commun. 125 (1998) 141.
[7] W.D. Furlan, G. Saavedra, J.A. Monsoriu, J.D. Patrignani, J. Opt. A: Pure Appl. Opt. 5 (2003) S361.
[8] J. Durnin Jr., Miceli, J.H. Eberly, Phys. Rev. Lett. 58 (1987) 1499.
[9] T. Eižmár, V. Garcés-Chávez, K. Dholakia, P. Zemánek, Appl. Phys. Lett. 86 (2005) 174101.
[10] J.Y. Lu, J. Cheng, Y. Li, B.D. Cameron, Proc. SPIE 4619 (2002) 300.
[11] T. Wulle, S. Herminghaus, Phys. Rev. Lett. 70 (1993) 1401.
[12] Z. Jaroszewicz, Axicons, Design and Propagation Properties, SPIE Polish Chapter Research and Development Series, Vol. 5, 1997.
[13] G. Indebetouw, J. Opt. Soc. Am. A 6 (1989) 150.
[14] A. Vasara, J. Turunen, A.T. Friberg, J. Opt. Soc. Am. A 6 (1989) 1748.
[15] Z. Jaroszewicz, V. Climent, V. Duran, J. Lancis, A. Kolodziejczyk, A. Burvall, A.T. Friberg, J. Mod. Opt. 51 (2004) 2185.
[16] C.J. Zapata-Rodríguez, F.E. Hernández, Opt. Commun. 254 (2005) 3.
[17] Y. Lin, W. Seka, J.H. Eberly, H. Huang, D.L. Brown, Appl. Opt. 31 (1992) 2708.
[18] J.H. McLeod, J. Opt. Soc. Am. 44 (1954) 592.
[19] A.G. Sedukhin, J. Opt. Soc. Am. A 17 (2000) 1059.
[20] M. Born, E. Wolf, Principles of Optics, 7th Edition, Chapter IV, Cambridge, University Press, 1999.
[21] C.J. Zapata-Rodríguez, Optik 113 (2002) 361.