Vortex Transmutation

Albert Ferrando,[1,2,*] Mario Zacarés,[3,*] Miguel-Ángel García-March,[1,2,*]
Juan A. Monsoriu,[4,*] and Pedro Fernández de Córdoba[2,*]
[1]Departament d'Òptica, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot (València), Spain
[2]Departamento de Matemática Aplicada and IMPA-UPV, Universidad Politécnica de Valencia, Camino de Vera, s/n, E-46022 Valencia, Spain
[3]Instituto de Ingeniería Energética, Universidad Politécnica de Valencia, Camino de Vera, s/n, E-46022 Valencia, Spain
[4]Departamento de Física Aplicada, Universidad Politécnica de Valencia, Camino de Vera, s/n, E-46022 Valencia, Spain
(Received 4 May 2005; published 12 September 2005)

Using group theory arguments and numerical simulations, we demonstrate the possibility of changing the vorticity or topological charge of an individual vortex by means of the action of a system possessing a discrete rotational symmetry of finite order. We establish on theoretical grounds a "transmutation pass rule" determining the conditions for this phenomenon to occur and numerically analyze it in the context of two-dimensional optical lattices. An analogous approach is applicable to the problems of Bose-Einstein condensates in periodic potentials.

DOI: 10.1103/PhysRevLett.95.123901 PACS numbers: 42.65.Tg, 03.75.Lm, 42.70.Qs

Vortices are a physical phenomenon common to all complex waves. Defined by a phase singularity implying the vanishing of the wave amplitude, their presence is ubiquitous in physics where examples of vortices can be found in as diverse systems as quantized superfluids and superconductors, Bose-Einstein condensates (BEC's), nonlinear optical structures, or low dimensional condensed matter or particle systems (for a review see [1,2]). The possibility of changing vortex properties using periodic systems is a natural step based on the well-known example of the different behavior of electrons with or without the presence of a crystal. Like electrons, the properties of vortices in a lattice have been shown to be qualitatively different from those in a homogeneous medium. Vortices have been numerically predicted to exist in two-dimensional (2D) arrays of coupled waveguides [3], in 2D periodic dielectric media with Kerr nonlinearities—and, equivalently, in 2D BEC's with periodic potentials—[4,5], and in photonic crystal fibers with defects [6]. They have been experimentally observed in optically-induced square photonic lattices [7,8]. In all cases, the periodic medium has a strong influence on vortex features thus opening the possibilities for their external manipulation. In this Letter, we will show how the vorticity or topological charge of a vortex—its most fundamental feature—can be manipulated by means of an external system that possesses a discrete rotational symmetry.

Angular momentum is conserved in a nonlinear medium with $O(2)$ rotational symmetry in the x-y plane described by a first-order evolution equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$ for the complex scalar field ϕ. If we consider a solution with well-defined angular momentum $l \in \mathbb{Z}$ [i.e., an eigenfunction of the angular momentum operator $-i\partial/\partial\theta$: $\phi_l = e^{il\theta} f_l(r)$] at a given axial point z_0, its evolution will preserve the value of l for all z. In a system possessing a discrete point-symmetry (described by the C_n and C_{nv} groups) the angular momentum is no longer conserved. However, in this case one can define another quantity $m \in \mathbb{Z}$, the Bloch or pseudoangular momentum, which is conserved during propagation [9]. The pseudoangular momentum m plays then the role of l in a system with discrete rotational symmetry. From the group theory point of view, the angular and pseudoangular momenta l and m are also the indices of the 2D irreducible representations of $O(2)$ and C_n, respectively [10–12]. Unlike l, the values of m are limited by the order of the point-symmetry group C_n: $|m| \leq n/2$ [9,12].

The appearance of this upper bound for the pseudoangular momentum m opens the interesting question of determining the behavior of solutions propagating in an $O(2)$ rotational invariant medium with well-defined angular momentum l after impinging a medium with discrete symmetry of finite order in which the value of l exceeds the upper bound for pseudoangular momentum. This question can be analyzed in the light of group theory. Let us consider a wave propagating in an $O(2)$ nonlinear medium corresponding to a solution ϕ_l (not necessarily stationary) with well-defined angular momentum l launched into a second nonlinear medium characterized by the C_n group. The surface separating the two media defines an $O(2)$-C_n interface that we locate at $z = 0$. We assume that its evolution is first order in z: $L(|\phi|)\phi = -i\partial\phi/\partial z$. Its evolution in the second medium is thus fully determined by the initial condition $\phi_l(0)$. The initial field $\phi_l(0)$ will excite a different representation of the C_n group depending on the value of l. Once this second wave is excited, it will propagate in the C_n medium by preserving its representation—defined by its pseudoangular momentum m. Let φ_m be a function in the representation of C_n characterized by index m. Let us determine now which values of l are allowed by symmetry to produce a nonzero projection of $\phi_l(0)$ onto φ_m for a given value of m. The projection coefficient is given by $c_{ml} = \int_{\mathbb{R}^2} \varphi_m^*(r,\theta)\phi_l(r,\theta;0)$. Since φ_m and ϕ_l belong to representations of C_n and $O(2)$, respectively, they both properly transform under a discrete rotation of order n: $\varphi_m(r,\theta+2\pi/n) = e^{im(2\pi/n)}\varphi_m(r,\theta)$ and $\phi_l(r,\theta+2\pi/n) = e^{il(2\pi/n)}\phi_l(r,\theta)$. Thus, by performing the change of variable $\theta \to \theta + 2\pi/n$ in the definition of c_{ml} one arrives at the symmetry relation $c_{ml} = \exp[i(l-m)2\pi/n]c_{ml}$. The c_{ml} coefficient is then zero unless the following condition is fulfilled:

$$l - m = kn \ (k \in \mathbb{Z}), \quad \text{where } |m| \leq \frac{n}{2}. \tag{1}$$

The m representation of C_n is thus excited by initial fields having angular momenta $l = m, m \pm n, m \pm 2n, \ldots$. In an $O(2)$-$O(2)$ interface each representation of angular momentum m in the second medium is excited by one, and only one, angular momentum component l arising from the first medium and verifying $l = m$. Contrarily, in an $O(2)$-C_n interface the symmetry restriction (1) implies that, due to the cutoff $|m| \leq n/2$ in the C_n medium, there are infinite angular momenta l that can excite a given representation of index m in the second medium.

Thus, when the incident field carries an angular momentum l that overcomes the limiting value for pseudoangular momentum in the second medium, it will excite a wave that will propagate with *different* constant pseudoangular momentum m given by the "pass rule" (1). This result is valid for waves verifying an equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$, linear or nonlinear, stationary or evolving. A particularly interesting situation occurs when the incident field is a vortex field of the $O(2)$ nonlinear medium. This vortex field ϕ_l^{v} is a stationary solution of the evolution equation with well-defined angular momentum $l \neq 0$: $L(|\phi_l^{\mathrm{v}}|)\phi_l^{\mathrm{v}} = -\mu\phi_l^{\mathrm{v}}$. We consider here individual "canonical" vortices with a single phase singularity (i.e., with only one point in which $\phi_l^{\mathrm{v}} = 0$): $\phi_l^{\mathrm{v}}(r,\theta,z) = e^{il\theta}f_l^{\mathrm{v}}(r)e^{-i\mu z}$ [13]. The vorticity or topological charge of such solutions will be given by the circulation of its phase gradient around the singularity $v = (1/2\pi)\oint \nabla \times \arg(\phi_l^{\mathrm{v}})d\mathbf{r}$, which equals the angular momentum for canonical vortices $v = l$. On the other hand, the propagating wave ϕ_m with pseudoangular momentum m excited by ϕ_l^{v} will evolve in the C_n medium. There are different options for the asymptotic states of ϕ_m when $z \to \infty$. One possibility is that this wave asymptotically tends to a stationary solution $\phi_m \xrightarrow{z\to\infty} \phi_m^{\mathrm{v}}(r,\theta,z) = e^{im\theta}g_m(r,\theta)e^{-i\mu' z}$ in the representation of C_n given by the conserved pseudoangular momentum m. If ϕ_m^{v} has a single phase singularity then it will have the structure of an individual canonical discrete-symmetry vortex, its vorticity or topological charge v' being directly given by m: $v' = m$ [12]. The formation in the C_n medium of an asymptotic stationary state in the form of a discrete-symmetry vortex is a dynamical issue that depends on the structural parameters of the second medium as well as on the characteristics of the input vortex field ϕ_l^{v}. If dynamics allows the stabilization of the discrete-symmetry vortex solution, the $O(2)$-C_n interface will realize the mapping of an $O(2)$ vortex with charge $v = l$ (exceeding the limiting value for pseudoangular momentum) into a C_n vortex with charge $v' = m \neq 0$, such that $v' < v$. The pass rule for pseudoangular momentum (1) becomes a pass rule relating input and output vorticities:

$$v - v' = kn \ (k \in \mathbb{Z}), \tag{2}$$

where v' presents a cutoff in terms of n given by: $|v'| < n/2$ (even n) and $|v'| \leq (n-1)/2$ (odd n) [12]. Note that $m = n/2$ solutions are not vortices but nodal or dipole-mode solitons [12]. We will refer to the process of mapping an individual vortex into another with different topological charge as "vortex transmutation."

We will provide now a physical example of a system in which the phenomenon of vortex transmutation takes place. It is an optical interface separating two 2D dielectric media with Kerr nonlinearity, these two media being a homogeneous medium and a 2D square optical lattice. This system is equivalent to a 2D BEC in which a periodic potential is abruptly switched on. They constitute an $O(2)$-C_4 interface given by the following equation:

$$(\nabla_t^2 - V(x,y,z) + \gamma(z)|\phi|^2)\phi = -i\frac{\partial\phi}{\partial z}, \tag{3}$$

in which ∇_t is the 2D gradient operator and $V(x,y,z) = V_0 + \theta(z)[V_1(x,y) - V_0]$ where $\theta(z)$ is the step function and V_0 and $V_1(x,y) = V_1[\cos^2(\frac{2\pi}{\Lambda}x) + \cos^2(\frac{2\pi}{\Lambda}y)]$ (V_1 is the potential strength and Λ is the lattice spatial period) define the refractive index profile of the homogeneous medium and of the 2D optical lattice, respectively: $V_0 = -(n^2 - n_0^2)$ and $V_1(\mathbf{x}) = -[n^2(\mathbf{x}) - n_0^2]$, n_0 being a reference refractive index introduced by the slowly varying envelope approximation. The nonlinear function $\gamma(z) = \gamma + (1-\gamma)\theta(z)$ permits the nonlinear response of the system to be different in the two media. All distances appearing in Eq. (3) are normalized and dimensionless ($\mathbf{x} = k_0\mathbf{x}'$, $z = k_0 z'$). In order to solve the evolution problem in this system we solve first Eq. (3) for $z < 0$, which becomes an ordinary nonlinear Schrödinger equation (NLSE) for a homogeneous medium. Since our aim is to evidence the phenomenon of vortex transmutation we are interested in finding canonical vortex solitons of different charges in the homogeneous $O(2)$ medium: $\phi_l^{\mathrm{v}}(\mathbf{x},z) = e^{il\theta}f_l^{\mathrm{v}}(r)e^{-i\mu z}$. This can be done by standard methods. At a given value of l, a family of $O(2)$ vortices are found characterized by their power $P_l = \int_{\mathbb{R}^2}|\phi_l^{\mathrm{v}}|^2$ and their propagation constant μ, which are related through the relation $P_l(\mu)$. In the case of a Kerr nonlinearity, μ behaves as a scaling parameter and P_l is μ independent [1]. Once the vortex solution ϕ_l^{v} is found, it is taken as an initial solution for propagating it in the 2D optical lattice ($z > 0$): $\phi(\mathbf{x},0) = \phi_l^{\mathrm{v}}(\mathbf{x},0) = e^{il\theta}f_l^{\mathrm{v}}(r)$. Thus we solve Eq. (3) for $z > 0$, which becomes a NLSE with the periodic potential

$V_1(\mathbf{x})$ satisfying the previous initial condition. This is solved numerically using a standard split-step Fourier evolution method.

According to our previous symmetry arguments, the evolution of the ϕ wave for $z > 0$ has to occur in a way that the pass rule for angular momentum (1) is fulfilled. The $O(2)$ vortex soliton ϕ_l^{v} carrying angular momentum l will excite a propagating wave ϕ_m for $z > 0$ in a representation of C_4 with pseudoangular momentum m given by Eq. (1). Indeed, numerical evidence of this pass rule is obtained by analyzing the rotational symmetry of the evolving field. By construction, the input momentum is l since we choose the solution to be of the form $\phi_l^{\mathrm{v}}(\mathbf{x}, z) = e^{il\theta} f_l^{\mathrm{v}}(r) e^{-i\mu z}$ for $z \leq 0$. In order to check the symmetry properties of the solution for $z > 0$, we numerically evaluate the rotated field $\bar{\phi}(r, \theta, z) \equiv \phi(r, \theta + \pi/2, z)$ at every step in z and compare it to its unrotated value $\phi(r, \theta, z)$. If ϕ belongs to the m representation of C_4, $\phi(r, \theta + \pi/2, z) = e^{im\pi/2}\phi(r, \theta, z)$ and the ratio $\bar{\phi}/\phi$ will have a constant value for all $\mathbf{x} \in \mathbb{R}^2$ (with the exception of $\mathbf{x} = \mathbf{0}$, where rotations are ill defined) and $z > 0$: $\bar{\phi}/\phi = e^{im\pi/2}$. If this condition is satisfied, the value of m can be directly extracted from the numerical ratio $\bar{\phi}/\phi$. Indeed, the independence of the $\bar{\phi}/\phi$ ratio from transverse coordinates is numerically verified at every axial step, which permits the evaluation of m for different values of $z > 0$. Results are shown in Fig. 1. These results nicely confirm general condition (1).

Once the angular momentum pass rule is checked there persists still the question of the fake of the propagating wave in the C_4 medium. As predicted by the theory, m is numerically conserved during evolution. However, the asymptotic behavior of the ϕ_m evolving field can be very different depending on the parameters of the incident vortex field (its power P and its propagation constant μ) and of the characteristics of the periodic potential $V_1(\mathbf{x})$ (the potential strength V_1 and the lattice period Λ). Our interest lies in obtaining asymptotic stationary states which can be described as individual or canonical discrete-symmetry vortices. This condition implies that the asymptotic field has to present a single phase singularity. In other words, we want to exclude multivortex or cluster excitations. In order to achieve this feature, we enlarge the optical lattice (by increasing its period Λ) according to the size of the input vortex for increasing values of l. Thus, in our simulations Λ is fixed by l.

After performing many different simulations, we have indeed found numerical evidence of the vortex transmutation phenomenon. By playing with the input parameters P and μ and the lattice strength V_1 and period Λ, we have been able to find asymptotic stationary states $\phi_m^{\mathrm{v}} = e^{im\theta} g_m(r, \theta) e^{-i\mu' z}$ for different values of the input vorticity value $v = l$. The vorticity of the output field can be only $v' = \pm 1$ because of the vorticity cutoff for a C_4 system (recall that $m = \pm 2$ solutions are not vortices but nodal or dipole-mode solitons [12]). In Fig. 2 we show the amplitudes and phases of input and output vortices for different input vorticity values v. All of them verify the vorticity pass rule (2). The vortex transmutation phenomenon only occurs when $|v| > 2$. Similar results are found for the corresponding input antivortices with negative values of v. When, for fixed $v = l$ (fixed Λ), the selection of P, μ, and V_1 is not adequate, the asymptotic solution can be nonstationary. We observe two different scenarios besides the stationary regime: discrete diffraction of the input wave in the optical lattice and self-focusing instability leading to filamentation of the field. A thorough analysis of multiple configurations permits to elaborate a vortex transmutation phase diagram where the three different regimes can be

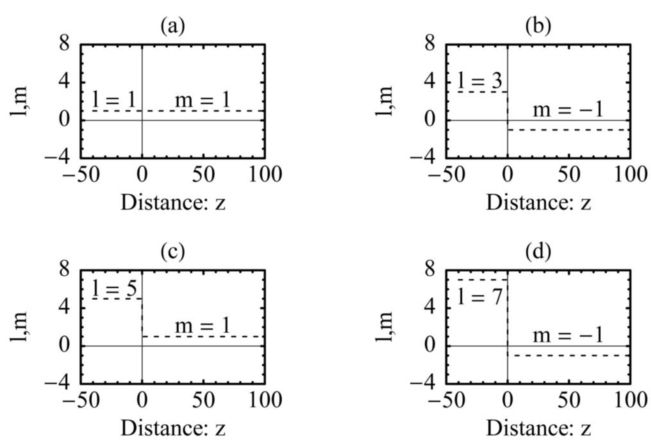

FIG. 1. Numerical confirmation of the angular momentum pass rule for different values of the angular momentum l of the incident vortex field: (a) $l = 1$ and $m = 1$; (b) $l = 3$ and $m = -1$; (c) $l = 5$ and $m = 1$; (d) $l = 7$ and $m = -1$.

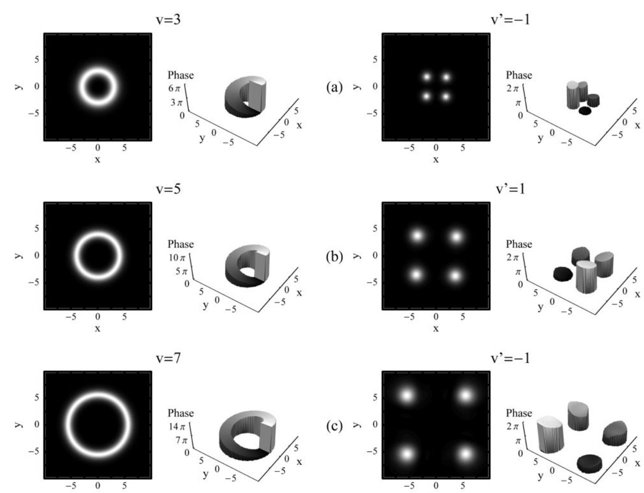

FIG. 2. Amplitudes and phases of input $O(2)$ vortices and output C_4 vortices for different values of the input vorticity v: (a) $v = 3$ and $v' = -1$ ($P = 2.5$, $\mu = 2$, $V_1 = 2$, $\Lambda = 1.3$); (b) $v = 5$ and $v' = 1$ ($P = 3.0$, $\mu = 3$, $V_1 = 3$, $\Lambda = 2.5$); (c) $v = 7$ and $v' = -1$ ($P = 3.5$, $\mu = 3$, $V_1 = 3$, $\Lambda = 3.5$).

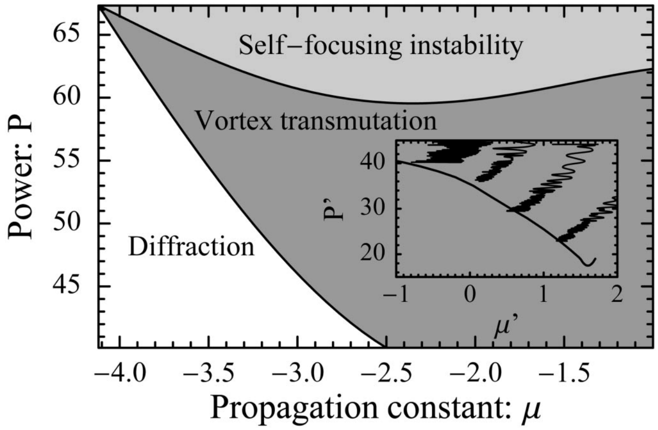

FIG. 3. Vortex transmutation phase diagram for $v = 3$ vortices in an optical lattice with $V_1 = 2$. Inset: Evolution in the (P', μ') plane of four initial $v = 3$ vortices characterized by different initial (P, μ) values. The solid line curve corresponds to the stationary $v' = -1$ vortices as in Ref. [14].

recognized. As an example, in the phase diagram shown in Fig. 3 we observe the vortex transmutation region (shaded) differentiated from the diffraction (white) and self-focusing instability (light shaded) regions as a function of power P and propagation constant μ of the input vortex field at fixed V_1. Analogous phase diagrams are found for different values of V_1. It is interesting to analyze the evolution of different vortex-transmuting configurations by monitoring the evolution of power $P'(z)$ and average propagation constant $\mu'(z) \equiv \int \phi^*(-i\partial/\partial z)\phi / \int \phi^* \phi$ (defined both on the finite domain of the numerical solution) in the C_4 medium. These quantities are z dependent, in general. However, they become independent of z when we analyze a stationary solution; thus we expect $[P'(z), \mu'(z)] \xrightarrow{z\to\infty} (P', \mu')$ for asymptotic stationary states. Every input $O(2)$ vortex characterized by the initial values (P, μ) defines then a different trajectory in the P'-μ' plane. Because of the presence of the interface, $-i\partial\phi/\partial z$ is discontinuous at $z = 0$, as one can check in Eq. (3). This fact implies that $\mu'(z = 0)$ has a different value from that of the input vortex μ. In any case, the initial μ value univocally defines $\mu'(z = 0)$ and, then, the trajectory in the second medium. In Fig. 3 (inset) we show four different trajectories mapping $O(2)$ $v = 3$ vortices with different (P, μ) initial values into asymptotic C_4 vortices with charge $v' = -1$ characterized by their (P', μ') values. It can be checked numerically that these values lie on the same $P'(\mu')$ curve found in Ref. [14] for stationary vortices with charge $v' = -1$ in an identical square optical lattice. By launching a whole family of initial vortices we have been able to asymptotically reproduce the entire $P'(\mu')$ curve of C_4 vortices. It is remarkable that the asymptotic C_4 vortices in the Fig. 3 inset have been checked to be stable under small perturbations [14] whereas the original $O(2)$ ones are not [1]. Thus the vortex transmutation phenomenon not only permits to change the charge of unstable input vortices but it can also help to transform them into stable structures. Transformation of optical vortices, including inversion of vortex charge, have been demonstrated and experimentally observed in ‘‘noncanonical’’ vortices in free space [13]. Here, however, all vortices involved are canonical, the key point for the vortex transmutation phenomenon to occur being the suitable matching between angular and pseudoangular momentum at the $O(2)$-C_n interface. Clearly the relation between angular and pseudoangular momentum reveals itself to be important for propagation in a discrete-symmetry medium, as shown in Ref. [9]. We would like to stress that the theory of transmuting vortices is general and applies to any system given by an equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$ in the presence of an $O(2)$-C_n interface. Hence, this phenomenon is expected to occur in a wide variety of physical systems as those mentioned in the introduction of this Letter.

This work was financially supported by Grant Nos. TIC2002-04527-C02-02 (MCyT and FEDER funds), GV04B-390, and Grupos03/227 (Generalitat Valenciana).

*Also at Interdisciplinary Modeling Group (InterTech)
Electronic address: http://www.upv.es/intertech

[1] A. S. Desyatnikov, L. Torner, and Y. S. Kivshar, nlin.PS/0501026 [Prog. Opt. (to be published)].
[2] *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1985).
[3] B. A. Malomed and P. G. Kevrekidis, Phys. Rev. E **64**, 026601 (2001).
[4] J. Yang and Z. H. Musslimani, Opt. Lett. **28**, 2094 (2003).
[5] B. Baizakov, B. A. Malomed, and M. Salerno, Europhys. Lett. **63**, 642 (2003).
[6] Ferrando *et al.*, Opt. Express **12**, 817 (2004).
[7] Neshev *et al.*, Phys. Rev. Lett. **92**, 123903 (2004).
[8] Fleischer *et al.*, Phys. Rev. Lett. **92**, 123904 (2004).
[9] A. Ferrando, nlin.PS/0411059.
[10] Ferrando *et al.*, Opt. Express **13**, 1072 (2005).
[11] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Addison-Wesley Series in Physics (Addison-Wesley, Reading, MA, 1964), 1st ed.
[12] A. Ferrando, M. Zacarés, and M. A. García-March, Phys. Rev. Lett. **95**, 043901 (2005).
[13] Molina-Terriza *et al.*, Phys. Rev. Lett. **87**, 023902 (2001); A. Ya. Bekshaev, M. S. Soskin, and M. V. Vasnetsov, J. Opt. Soc. Am. A **20**, 1635 (2003).
[14] Z. H. Musslimani and J. Yang, J. Opt. Soc. Am. B **21**, 973 (2004).