

Nodal solitons and the nonlinear breaking of discrete symmetry

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Abstract: We present a new type of soliton solutions in nonlinear photonic systems with discrete point-symmetry. These solitons have their origin in a novel mechanism of breaking of discrete symmetry by the presence of nonlinearities. These so-called nodal solitons are characterized by nodal lines determined by the discrete symmetry of the system. Our physical realization of such a system is a 2D nonlinear photonic crystal fiber owning \mathcal{C}_{6v} symmetry.

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1. Introduction

Symmetry is one of the most powerful and elegant concepts in physics. The use of group theory provides an extraordinary mathematical tool to classify solutions according to the symmetries of the physical system. In recent years, the increasing interest in physical systems owning 2D discrete symmetries, such as 2D nonlinear photonic crystals or Bose-Einstein condensates in 2D periodic potentials, raises the question of utilizing group theory as an analysis tool. Certainly, this approach is a standard in solid state physics. Its use in the topic of photonic crystals is less extended and it has been traditionally confined to the classification of linear modes [1]. Its generalization to the nonlinear case has become of great interest after the recent experimental observation of fundamental and vortex solitons in optically-induced 2D nonlinear photonic crystals [2, 3, 4]. In fact, attempts to apply group theory to the analysis of this type of solutions permitted the analytical prediction of the angular dependence of vortex solitons in 2D photonic crystals [5]. Following this approach, in this paper we will use group theory as a general framework to analyze the role played by nonlinearities in the realization of discrete symmetry. We will see that discrete symmetry is realized differently when nonlinearities are present and that a new phenomenon of discrete-symmetry breaking occurs. The physical outcome of this general process is the generation of a new type of solitons with lesser symmetry than that of the original system.

2. Group self-consistency condition

So that, we start by analyzing the general problem of finding stationary solutions $-\Phi(x, y, z) = \phi(x, y) \exp i\beta z$ of a nonlinear operator of the form:

$$(L_0 + L_{NL}(|\Phi|)) \Phi = -\frac{\partial^2 \Phi}{\partial z^2}, \quad (1)$$

where L_0 is a linear operator (depending on the transverse coordinates $\mathbf{x}_t = (x, y)$ only) invariant under a 2D discrete point-symmetry group G and L_{NL} is a nonlinear operator depending locally on the modulus of the ϕ field. We are interested then in solving the following nonlinear eigenvalue problem:

$$(L_0 + L_{NL}(|\phi|)) \phi = \beta^2 \phi. \quad (2)$$

Since L_0 is such that $[L_0, G] = 0$ (i.e., it is invariant under the action of all the elements of the group G : $g^{-1}L_0g = L_0, \forall g \in G$), all its eigenmodes have to lie on finite representations of the discrete group G [6]. What we try to determine next is the effect of the nonlinear term L_{NL} in the symmetry properties of Eq. (2) solutions.

A solution ϕ_s of Eq. (2) has to satisfy the so-called self-consistency condition, namely, ϕ_s has to appear as an eigenmode of the operator generated by itself, $L(\phi_s) \equiv L_0 + L_{NL}(|\phi_s|)$. From a symmetry point of view, the self-consistency condition implies that if ϕ_s belongs to some representation of a finite group G' , then the entire operator $L(\phi_s)$ has to be invariant under the same group, $[L(\phi_s), G'] = 0$ (otherwise, $L(\phi_s)$ would not contain in its spectrum the representation where ϕ_s lies on). We call this property the group self-consistency condition.

The simplest attempt to find solutions of Eq. (2) satisfying the group self-consistency condition is trying functions that enjoy the full symmetry of the linear operator; i.e., functions that are invariant under G . Functions belonging to the fundamental representation of G satisfy this property [6]: $\phi_{\text{fund}}^g \equiv g\phi_{\text{fund}} = \phi_{\text{fund}}, \forall g \in G$. Group self-consistency is satisfied because $g^{-1}L_{NL}g = L_{NL}(|\phi_{\text{fund}}^g|) = L_{NL}(|\phi_{\text{fund}}|), \forall g \in G$; i.e., $[L_{NL}, G] = 0$ and, thus, $[L(\phi_{\text{fund}}), G] = 0$. Solutions that satisfy this property are called fundamental solitons and they have been found in different systems of the type described by Eq. (2). A less obvious choice is the selection of functions belonging to higher-order representations of the same symmetry group G of the

linear system. For 2D point-symmetry groups, these higher-order representations can be either non-degenerated (one-dimensional) or doubly-degenerated (two-dimensional) [6]. Vortex-antivortex solutions, appearing always as conjugated pairs (ϕ_v, ϕ_v^*) , belong to two-dimensional representations of G [5]. We note that the modulus of a vortex solution is a group invariant (this is a general property also fulfilled by the modulus of functions belonging to one-dimensional representations of G). Since $|\phi_v^G| = |\phi_v|$, then $g^{-1}L_{NL}g = L_{NL}(|\phi_v^g|) = L_{NL}(|\phi_v|)$, $\forall g \in G$ and, consequently, $[L_{NL}, G] = 0$ and $[L(\phi_v), G] = 0$. It is apparent that vortex solitons also fulfill the group self-consistency condition.

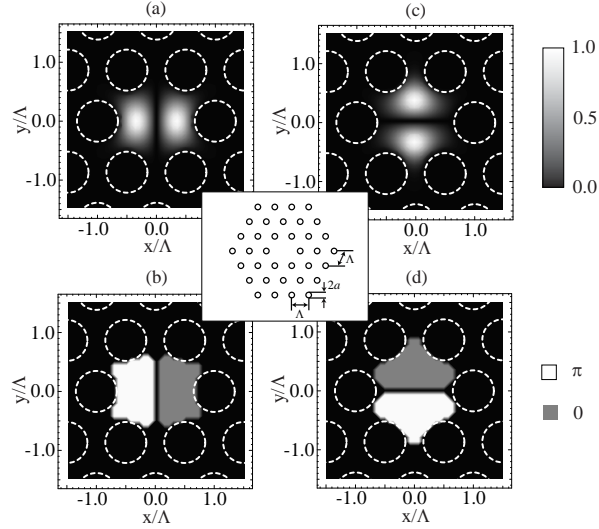


Fig. 1. Two nodal solitons for $l = 1$ ($\Lambda = 23 \mu\text{m}$, $a = 8 \mu\text{m}$, $\gamma = 0.006$ and wavelength $\lambda = 1064 \text{nm}$): (a)-(b) amplitude and phase, respectively, of the S nodal soliton; (c)-(d) amplitude and phase, respectively, of the A nodal soliton. Inset: schematic transverse representation of a PCF.

3. Group self-consistency theorem and nonlinear breaking of discrete symmetry

Up to now, we have considered solutions belonging to representations of G , the symmetry group of the linear operator. For them, the self-induced nonlinear operator enjoys the same symmetry as its linear counterpart: $[L_{NL}(|\phi_s|), G] = [L_0, G] = 0$. However, we ask ourselves if solutions with different symmetry than that exhibited by G can also fulfill the group self-consistency condition. Let us assume a trial function belonging to a certain representation of a group G' such that $G' \neq G$. Since the modulus of the function is G' -invariant, $|\phi_s^{G'}| = |\phi_s|$, the nonlinear operator is also G' -invariant, $[L_{NL}(|\phi_s|), G'] = 0$. If $G' > G$, the total operator $L(\phi_s)$ cannot have the G' symmetry because the linear operator has less symmetry. The linear part of $L(\phi_s)$ breaks the G' symmetry of the nonlinear part and the group self-consistency condition cannot be satisfied: $[L(\phi_s), G'] \neq 0$. Thus, we disregard functions with symmetry higher than G . This scenario changes if one considers functions with lesser symmetry than G ; more specifically, functions belonging to representations of a subgroup G' of G ($G' \subset G$). The difference now with respect to the previous case is the following: since the linear part is invariant under the G group, $[L_0, G] = 0$, it is also invariant under any of its subgroups. Thus, it is also true that $[L_0, G'] = 0$. Since, as before, $|\phi_s^{G'}| = |\phi_s|$, the nonlinear operator verifies $[L_{NL}(|\phi_s|), G'] = 0$

and, consequently, the total operator is G' -invariant, $[L(\phi_s), G'] = 0$. The function then fulfills the group self-consistency condition for the subgroup G' . Therefore, this type of functions can also be solution of the eigenvalue Eq. (2) (note that solutions with no symmetry also verify the group self-consistency condition since the identity transformation constitutes a subgroup of any group G). We cannot guarantee that they are indeed solutions until we solve Eq. (2) with the constraint $\phi \in D(G')$ ($D(G')$ being a representation of G'), since the trivial solution ($\phi = 0$) is always valid. In this sense, group self-consistency is a necessary but not sufficient condition.

We can summarize the previous results in a single statement in the form of what we call the *group self-consistency theorem*: if a system described by Eq. (2) is invariant under some discrete-symmetry group G then any of its solutions either belongs to one representation of the group G or to one of its subgroups G' ($G' \subset G$). Note that this theorem is a consequence of the group self-consistency condition and that it has strong implications on the symmetry features of the amplitude and phase of permitted solutions, as it was shown for optical vortices in Ref. [5]. It should be emphasized the importance of knowing that allowed solutions lie in a well-defined representation of a given group. This fact implies that symmetry properties of amplitudes and phases of permitted solutions are unequivocally fixed by strong symmetry constraints. Besides helping to understand the symmetry characteristics of solutions owning the same symmetry of the system, such as the aforementioned optical vortices, the *group self-consistency theorem* permits to predict the existence of new solutions characterized by a lesser symmetry than that of the system, as it will be shown next. From the symmetry point of view, these new solutions would be characterized by the symmetry breaking pattern $G \rightarrow G' \subset G$ which would distinguish them from previously reported soliton solutions owning the same symmetry of the system $G' = G$ [2][3][4][5][7]. In other words, we would say that the latter are symmetry-preserving solutions whereas the former are symmetry-breaking ones, for which nonlinearity breaks the discrete symmetry of the optical system.

4. Nodal solitons in photonic crystal fibers

Our specific physical system is a triangular photonic crystal fiber (PCF), although similar results can be expected in other 2D photonic crystals. We study the propagation of the electric component of a monochromatic electromagnetic field (at fixed polarization: $\mathbf{E} = \phi \mathbf{u}$, $|\mathbf{u}| = 1$). PCF's are thin silica fibers possessing a regular array of holes extending the entire fiber length and characterized by the hole radius a and the spatial period Λ of the photonic crystal cladding (see inset in Fig. 1). When silica nonlinearity is not neglected, a PCF is a particular case of a 2D nonlinear photonic crystal with a defect (where guidance occurs). In this case, $L_0 = \nabla_t^2 + k_0^2 n_0^2(x, y)$, where ∇_t is the transverse gradient operator, k_0 is the vacuum wave number, and n_0 is the refractive-index profile function ($n_0 = n_{(\text{silica})}$ in silica and $n_0 = 1$ in air). The nonlinear term is $L_{NL} = k_0^2 \gamma \Delta(x, y) |\phi|^2$, where Δ is the distribution function of nonlinear material ($\Delta = 1$ in silica and $\Delta = 0$ in air) and γ is a dimensionless nonlinear coupling constant, $\gamma \equiv 3\chi_{(\text{silica})}^{(3)} P / (2\epsilon_0 c n_{(\text{silica})} A_0)$ (P is the total power and A_0 is an area parameter: $A_0 = \pi(\Lambda/2)^2$).

The symmetry group of a triangular PCF is \mathcal{C}_{6v} , i.e., $[L_0, \mathcal{C}_{6v}] = 0$. This group is constituted by discrete $\pi/3$ -rotations ($R_{\pi/3}$) plus specular reflections with respect to the x and y axes: $\theta \xrightarrow{R_x} -\theta$ and $\theta \xrightarrow{R_y} \pi - \theta$, in polar coordinates. Solutions with the PCF \mathcal{C}_{6v} -symmetry have been previously found in the form of fundamental and vortex solitons [7, 5]. We will focus now on new solutions belonging to the subgroup \mathcal{C}_{2v} of \mathcal{C}_{6v} , formed by R_{π} , R_x and R_y . Thus, we study the particular symmetry breaking pattern $G = \mathcal{C}_{6v} \rightarrow G' = \mathcal{C}_{2v}$. We can explicitly construct functions belonging to the four non-degenerated representations of \mathcal{C}_{2v} out of functions in the two-dimensional representations of \mathcal{C}_{6v} . The latter functions come in conjugated pairs (ϕ_l, ϕ_l^*)

($l = 1, 2$), whose angular dependence is fixed by symmetry: $\phi_l = r^l e^{il\theta} \phi_l^s(r, \theta) \exp[i\phi_l^p(r, \theta)]$, where $\phi^s(r, \theta)$ is a scalar function, characterized by $\phi^s(r, \theta + \pi/3) = \phi^s(r, \theta)$ and $\phi^s(r, -\theta) = \phi^s(r, \pi - \theta) = \phi^s(r, \theta)$, and $\phi^p(r, \theta)$ is a pseudo-scalar function characterized by $\phi^p(r, \theta + \pi/3) = \phi^p(r, \theta)$ and $\phi^p(r, -\theta) = \phi^p(r, \pi - \theta) = -\phi^p(r, \theta)$. Let us consider the two following linear combinations ($l = 1, 2$): $1/\sqrt{2} [\phi_l \pm \phi_l^*]$. By writing the angular dependence of ϕ_l , the new functions adopt the form:

$$\phi_\delta^l(r, \theta) = \sqrt{2} r \phi_l^s(r, \theta) \cos [l\theta + \phi_l^p(r, \theta) + \delta], l = 1, 2. \quad (3)$$

where $\delta = 0, \pi/2$. The four different type of solutions given by Eq. (3) belong to the four different one-dimensional representations of the \mathcal{C}_{2v} group; i.e., in Hamermesh's notation: $\phi_0^1 \in B_1, \phi_{\pi/2}^1 \in B_2, \phi_0^2 \in A_1, \phi_{\pi/2}^2 \in A_2$. According to our previous general argument, the ϕ_δ^l functions can be solutions of Eq. (2) for a PCF. Therefore, we solve Eq. (2) with the ansatz given by Eq. (3) by means of the Fourier iterative method previously used in Refs. [7] and [5] to find fundamental and vortex soliton solutions in PCF's. This method ensures that the group representation is preserved in the iterative process. Starting from a seed function of the form (3), the method either finds the trivial solution or converges to a solution belonging to one of the representations of \mathcal{C}_{2v} .

Solutions of Eq. (2) of the form given by Eq. (3) are indeed found. They are characterized by nodal lines determined by symmetry through the implicit equation $\cos [l\theta + \phi_l^p(r, \theta) + \delta] = 0$ ($l = 1, 2$). For this reason, we call them nodal solitons. The solution with $\delta = 0$ corresponds to the symmetric (S) \mathcal{C}_{6v} combination $\phi_S^l \equiv 1/\sqrt{2}(\phi_l + \phi_l^*)$ and that with $\delta = \pi/2$ to the antisymmetric (A) one $\phi_A^l \equiv i/\sqrt{2}(\phi_l - \phi_l^*)$ ($l = 1, 2$). Note that the ϕ_l function is not a solution of the Eq. (2) because the superposition principle does not hold. This function can only be approximated by a vortex solution in the linear regime ($\gamma \approx 0$). Although the complete structure of nodal lines could be rather intricate, S and A nodal solitons are characterized by principal nodal lines: a single principal line for $l = 1$ solitons and two orthogonal principal lines for $l = 2$. In our simulations, we have found this four different types of nodal solitons. Nevertheless, we will show here results corresponding to the S and A nodal soliton solutions with $l = 1$ only. In Fig. 1 we show the amplitude and phase of S and A nodal solitons, respectively. As predicted by the nodal line condition, the $l = 1, S$ nodal soliton presents a single vertical nodal line, whereas for the A soliton this line is horizontal.

At this point, it is interesting to remark that the so-called "dipole lattice solitons" found in perfectly periodic \mathcal{C}_{4v} lattices [8] can be analogously explained in this framework. They are nothing but nodal solitons associated to the breaking of the original \mathcal{C}_{4v} symmetry of the photonic crystal into its subgroup \mathcal{C}_2 . They correspond to the symmetry-breaking pattern $\mathcal{C}_{4v} \rightarrow \mathcal{C}_2$ and, consequently they belong to the symmetric (S) and antisymmetric (A) representations of the \mathcal{C}_2 group. The numerically-found amplitudes and phases of "dipole lattice solitons" are exactly those predicted by group theory.

5. The role of symmetry. Soliton spectrum and stability

The role played by symmetries can be clearly envisaged by analyzing the diagram $n_{\text{sol}} \text{ vs. } \gamma$, where $n_{\text{sol}} = \beta/k_0$ for different soliton solutions. In Fig. 2 we represent the curves corresponding to the S and A nodal solitons as well as the curves for vortex solitons reported in Ref. [5]. In the linear case ($\gamma = 0$), the superposition principle holds and, therefore, the symmetric and antisymmetric combinations of the linear modes ϕ_1 and ϕ_1^* (the linear modes of vortex-type with $l = 1$) are degenerated solutions. In the linear case, all of them (S, A, ϕ_1 and ϕ_1^* modes) belong to the same representation of the \mathcal{C}_{6v} group ($l = 1$, or E_2 in Hamermesh's notation) and, for this reason, they all have the same effective index. The presence of the nonlinearity changes this scenario. It provides different options for the realization of the discrete symmetry.

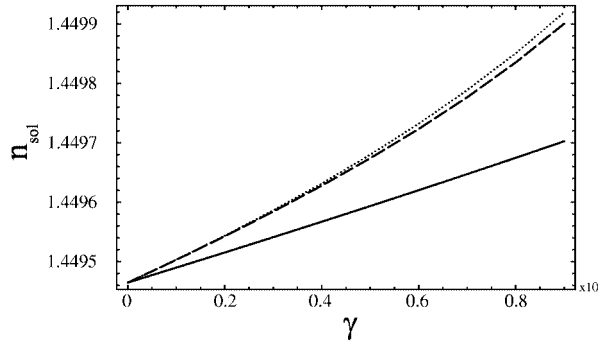


Fig. 2. Effective index of a soliton solution, n_{sol} vs the nonlinear coupling γ for symmetric (dotted line) and antisymmetric (dashed line) solitons and vortex and antivortex solitons with $l = 1$ (solid line).

Vortex solitons realize the discrete symmetry of the linear system, $[L(|\phi_1\rangle), \mathcal{C}_{6v}] = 0$, whereas nodal solitons break this symmetry into its \mathcal{C}_{2v} subgroup, $[L(|\phi_{S,A}\rangle), \mathcal{C}_{2v}] = 0$. Consequently, their corresponding eigenvalues are different since they are no longer related by the original symmetry that they enjoyed in the linear case ($\gamma = 0$). Moreover, group theory predicts that the vortex and antivortex solutions must have the same effective index as they provide the same total operator $L(|\phi_1\rangle)$ and they belong to the same representation (E_2) of it. This is not the case for nodal solitons. ϕ_S and ϕ_A are not in the same representation of \mathcal{C}_{2v} . This fact immediately implies that its corresponding eigenvalues must be different. Curves in Fig. 2 explicitly manifest this feature. For small nonlinearities, all curves appear as nearly-degenerated, but as the value of γ increases, a growing gap between nodal solitons and vortices occurs. Even larger values of γ permit to manifest the existing gap between the S and A nodal solitons.

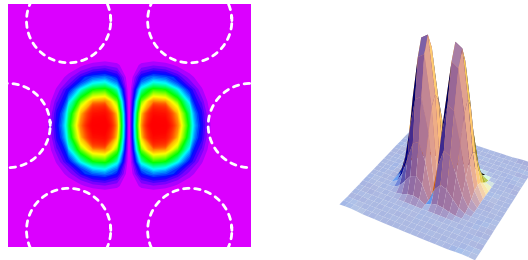


Fig. 3. Evolution of a diagonal perturbation of a S nodal soliton showing asymptotic stability. (749 KB)

In order to check the stability of nodal solitons we need to solve the evolution Eq. (1) after perturbing the solution: $\phi'_{S,A} = \phi_{S,A} + \delta\phi$. The stability analysis follows that of vortex solitons in Ref. [5]. In there we introduced the concepts of diagonal and non-diagonal perturbations. For one-dimensional representations, a diagonal perturbation is defined as that that preserves the representation in which the solution lies on. In the present case—in which ϕ_S and ϕ_A belongs to the B_1 and B_2 one-dimensional representations of \mathcal{C}_{2v} , respectively—, this definition implies that $\phi'_S \in B_1$ and $\phi'_A \in B_2$. Explicit examples of such perturbations are scaled solutions: $\phi'_{S,A} = (1 + \varepsilon)\phi_{S,A}$, $\varepsilon \neq 0$. Evolution yields numerical evidence that nodal solitons are stable under such perturbations, as shown in Fig. 3 in which a diagonal perturbation (scaled solution)

is applied. However, non-diagonal perturbations, taking the perturbed solution out of its original representation, provide instabilities. These instabilities are of the oscillatory type, as shown in Fig. 4, and they can be understood as a simultaneous oscillation among modes belonging to all the representations of \mathcal{C}_{2v} . This instability pattern, however, shows no trace of pseudo-soliton collapse nor of transverse ejection of pseudo-solitons typical of Kerr nonlinearities in homogeneous media treated in the paraxial approximation. This particular behavior was first observed in vortices in PCF's [5]. Since the self-focussing instability seems to be rooted in the paraxial approximation [9], a plausible explanation of its absence is the non-paraxial nature of evolution in this case. Absence of ejection can be qualitative understood by the inhibition of transverse radiation induced by the photonic crystal cladding.

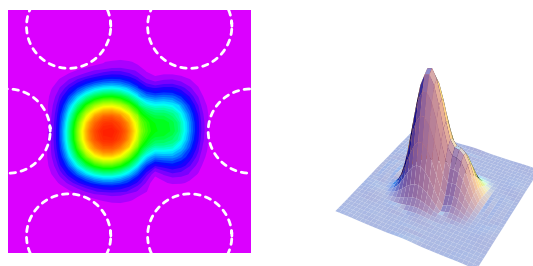


Fig. 4. Non-diagonal perturbation of a S nodal soliton. In this case, an oscillatory instability occurs. (576 KB)

An interesting interpretation of nodal solitons is as interacting pseudo-solitons. It can be proven that a S nodal soliton can be written as $\phi_S = \phi_0(x + x_0, y) - \phi_0(x - x_0, y)$, ϕ_0 being a localized function in the fundamental representation of \mathcal{C}_{6v} . In the case that ϕ_0 is a sufficiently localized function (large nonlinear coupling γ or strong lattice index contrast), ϕ_0 can be approximated by a fundamental soliton solution. Then, a nodal soliton can be envisaged as a pair of weakly interacting pseudo-solitons. Like in an homogeneous medium this interaction is repulsive [10]. In our case, nor γ nor the index contrast are necessarily large, consequently, the soliton-soliton interaction cannot longer be considered weak. However, the group theory anti-symmetric decomposition in terms of localized solutions in the fundamental representation of \mathcal{C}_{6v} remains valid. In this way, the concept of nodal soliton generalizes the idea of interacting pseudo-solitons into a regime of strong particle coupling (intermediate γ and index contrast). In the weak soliton-interaction regime, new soliton solutions have been recently found that can also be interpreted in the context of group theory reported in this paper [11].

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