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# Upper bounds of the first eigenvalue of closed hypersurfaces by the quotient area/volume

By

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**Abstract.** In this paper we obtain, for compact hypersurfaces $M$ embedded into Hadamard manifolds, an upper sharp bound of the first closed eigenvalue. This bound depends on the isoperimetric quotient $\text{Volume}(M)/\text{Volume}(\Omega)$, where $\Omega$ is the domain enclosed by $M$. More precise bounds are given when the ambient space is the complex or quaternionic hyperbolic space.

**1. Introduction.** After the work of Bleecker and Weiner ([5]), where these authors obtained an upper bound of the first closed eigenvalue $\lambda_1(M)$ of a closed (compact without boundary) submanifold $M$ of $\mathbb{R}^n$ which depends only on the volume, the dimension of $M$ and the integral along $M$ of the square of the norm of its second fundamental form, it is now a classical subject the obtaining of upper bounds of $\lambda_1$ which depend only on the volume, the dimension of $M$ and other extrinsic invariant of $M$. As an example are the papers by Reilly [19], Chen [7], Ros [21], Heintze [14], El Soufi and Ilias [9], Veeravalli [22] and Grosjean [13], where these extrinsic invariants are integrals of functions of the mean curvature or higher order mean curvatures of $M$. Sometimes (especially in [22]) some integrals of the distance to some center of mass are also involved in the upper bound. In this paper we obtain, for embedded hypersurfaces, an upper sharp bound by another extrinsic invariant: the volume enclosed by $M$. Concretely, we shall prove:

**Theorem 1.** *Let $M$ be an embedded closed hypersurface of a $n$-dimensional simply connected Riemannian manifold $\mathcal{M}$ with sectional curvature $K_{\text{sec}} \leqq 0$, and let $\Omega$ be the compact domain bounded by $M$, then*

$$\lambda_1(M) \leqq \frac{n-1}{n^2}\left(\frac{\text{Volume}(M)}{\text{Volume}(\Omega)}\right)^2, \tag{1.1}$$

*and the equality holds if and only if $\Omega$ is a geodesic ball of the euclidean space.*

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Let us remark that, since $\mathcal{M}$ is simply connected and $K_{\text{sec}} \leqq 0$, then it is diffeomorphic to $\mathbb{R}^n$, then every closed (compact without boundary) hypersurface $M$ is orientable, then it is the boundary of a compact domain.

Theorem 1 extends to manifolds with $K_{\text{sec}} \leqq 0$ a result of Alencar, do Carmo and Rosenberg on $\mathbb{R}^n$ (case $r = 0$ of Th. 1.3 in [2]). Related work in this direction are the papers by Grosjean [12], Alencar, do Carmo and Marques [1] and Alías and Malacarne [3].

For the case $K_{\text{sec}} \leqq \lambda$, with $\lambda$ any real number, the complex hyperbolic space $\mathbb{C}H^n(\lambda)$ and the quaternionic hyperbolic space $\mathbb{Q}H^n(\lambda)$, it is possible to obtain a better bound for $\lambda_1$. In order to state these more refined results, we need the concept of $f$-center of mass.

Given a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$, *the $f$-center of mass* of a compact submanifold $M$ of a Riemannian manifold $B$ is the point of $B$ where the function $\mathcal{F} : B \longrightarrow \mathbb{R}$ defined by $\mathcal{F}(p) = \int_M f(r_p(x))dx$ (where $r_p$ is the distance to $p$ in $B$ and $dx$ is the volume element of $M$) attains its minimum value. It generalizes the usual center of mass as defined, for instance, in [6] and [15]. This $f$-center of mass (for concrete values of $f$) has been used before by several authors ([8], [10], [14], [22]).

We shall use the notation:

$$s_\lambda(t) := \begin{cases} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, & \text{if } \lambda > 0 \\ t, & \text{if } \lambda = 0 \\ \frac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}}, & \text{if } \lambda < 0 \end{cases},$$

$$c_\lambda(t) := {s_\lambda}'(t), \quad ta_\lambda(t) := \frac{s_\lambda(t)}{c_\lambda(t)}, \quad co_\lambda(t) := \frac{c_\lambda(t)}{s_\lambda(t)}.$$

These functions satisfy the following computational rules:

$$c_\lambda^2 + \lambda\, s_\lambda^2 = 1, \quad c_{4\lambda} = c_\lambda^2 - \lambda\, s_\lambda^2, \quad s_{4\lambda} = s_\lambda c_\lambda. \tag{1.2}$$

Theorem 1 is a particular case of the following

**Theorem 2.** *Let $M$ be an embedded closed hypersurface of a $n$-dimensional simply connected Riemannian manifold $\mathcal{M}$ with sectional curvature $K_{sec} \leqq \lambda$, and let $\Omega$ be the compact domain bounded by $M$. If $\lambda > 0$ suppose that $M$ is contained in a geodesic ball of radius minor than $\frac{\pi}{2\sqrt{\lambda}}$. Let $p$ be the $(t \mapsto \int_0^t \varphi_\lambda(s)ds)$-center of mass of $M$ and let $\rho_M$ be the minimal distance from $M$ to $p$.*

*If $\lambda \leqq 0$ then*

$$\lambda_1(M) \leqq (n-1)\left(\frac{\text{Volume}(M)}{\text{Volume}(\Omega)}\right)^2 \frac{\varphi_\lambda^2(\rho_M)}{s_\lambda^2(\rho_M)}, \tag{1.3}$$

*where*

$$\varphi_\lambda(t) := \frac{\int_0^t s_\lambda^{n-1}(s)ds}{s_\lambda^{n-1}(t)}.$$

*If $\lambda > 0$, only for $n = 2$ we obtain the same inequality, which has the form*

$$\lambda_1(M) \leqq \frac{1}{(1+\cos(\sqrt{\lambda}\rho_M))^2}\left(\frac{\text{Volume}(M)}{\text{Volume}(\Omega)}\right)^2. \tag{1.4}$$

*The equality in (1.3) and (1.4) holds if and only if $\Omega$ is a geodesic ball of the simply connected manifold with sectional curvature $\lambda$.*

**Theorem 3.** *Let $M$ be an embedded closed hypersurface of $\mathbb{C}H^n(\lambda)$. Let $p$ be the $\ln c_\lambda(t)$-center of mass of $M$. Let $\Omega$ be the compact domain bounded by $M$, and let $\rho_M$ be the minimal distance from $M$ to $p$. Then*

$$\lambda_1(M) \leqq \frac{1}{4n^2}\left(\frac{\text{Volume}(M)}{\text{Volume}(\Omega)}\right)^2 (2n-1+\lambda\, ta_\lambda^2(\rho_M))(1+\lambda\, ta_\lambda^2(\rho_M)), \tag{1.5}$$

*and the equality holds if and only if $\Omega$ is a geodesic ball in $\mathbb{C}H^n(\lambda)$.*

**Theorem 4.** *Let $M$ be an embedded closed hypersurface of $\mathbb{Q}H^n(\lambda)$. Let $p$ be the $\frac{1}{(4n+2)\lambda}(\frac{1}{4n}ta_\lambda^2(t) - \ln c_\lambda(t))$-center of mass of $M$. Let $\Omega$ be the compact domain bounded by $M$, and let $\rho_M$ be the minimal distance from $M$ to $p$. Then*

$$\lambda_1(M) \leqq \left(\frac{\text{Volume}(M)}{\text{Volume}(\Omega)}\right)^2 \left(\frac{4n\, c_\lambda^2(\rho_M)+2}{4n(4n+2)\, c_\lambda^3(\rho_M)}\right)^2 \left(\frac{3}{c_\lambda^2(\rho_M)} + 4n - 4\right) \tag{1.6}$$

*and the equality holds if and only if $\Omega$ is a geodesic ball in $\mathbb{Q}H^n(\lambda)$.*

Remark 1. With the obvious change of $n$ by $2n$ for $\mathbb{C}H^n(\lambda)$ and $n$ by $4n$ for $\mathbb{Q}H^n(\lambda)$, the expression (1.1), with the strict inequality, is still a sharp bound for the closed hypersurfaces of $H^n(\lambda)$, $\mathbb{C}H^n(\lambda)$ and $\mathbb{Q}H^n(\lambda)$, because the upper bound in (1.3), (1.5) and (1.6) is the $\lambda_1$ of a geodesic sphere of radius $\rho_M$ in $H^n(\lambda)$, $\mathbb{C}H^n(\lambda)$ and $\mathbb{Q}H^n(\lambda)$, and it goes to (1.1) when $\rho_M \to 0$.

Remark 2. In all the above theorems, the hypothesis "$M$ embedded" is used only for the existence of the domain $\Omega$ bounded by $M$. For immersed submanifolds, the same computations give sharp inequalities if we change Volume($\Omega$) by the integral expression of (2.4) in (1.3) and (1.4), the integral expression of (2.5) in (1.5), and the integral expression of (2.6) in (1.6).

Again the same computations work for Theorems 1 and 2 when $M$ is only a $k$-dimensional immersed submanifold (and $\lambda \leqq 0$). The corresponding sharp bounds are like (1.3), just adding the change of $n-1$ by $k$ to the modifications indicated above.

Remark 3. Along the proof of these theorems, we shall obtain more precise bounds for $\lambda_1$, those obtained by changing the functions of $\rho_M$ which appear in (1.3), (1.4), (1.5) and (1.6) by the integrals, along $M$, of the same functions of the distance to the center of mass, and changing Volume$(M)^2$ by Volume$(M)$.

Remark 4. In part ii) of the Theorem in [22], it is given a non sharp inequality for $\lambda_1(M)$ when $M$ is a hypersurface in the hyperbolic space. If we write the inequality (1.3) under the form indicated in the Remarks 2 and 3, it is a sharp bound for $\lambda_1(M)$ of the type looked for in [22].

Before proving the theorems, we shall state, in the next section, some results on the Laplacian of the distance, Volume$(\Omega)$ and the $f$-center of mass, which will be used in their proofs. The essential tool for the proving the theorems is an appropriate modification of the method developed by Heintze in [14], which, in tourn, is an approppriate modification of the method developed by Reilly in [19]. In Section 3 we shall give the complete detailed proof of Theorem 3, and we shall indicate the way that proof of Theorem 3 has to be modified to prove Theorems 2 and 4.

**2. Preliminaries.** From now on, $\mathcal{M}$ well be a simply connected Riemannian manifold with sectional curvature $K_{\sec} \leqq \lambda$. $\nabla$, $\Delta$ and grad will denote the covariant derivative, the Laplacian and the gradient (respectively) of the ambient manifold $\mathcal{M}$, $\mathbb{C}H^n(\lambda)$ or $\mathbb{Q}H^n(\lambda)$. For $\Delta$ we shall use the following sign:

$$\Delta f = -\mathrm{tr}\nabla^2 f.$$

The corresponding operators in $M$ will be denoted by $\nabla^M$, $\Delta_M$ and $\mathrm{grad}_M$.

Given any point $p$ in the ambient space $\mathcal{M}$, $\mathbb{C}H^n(\lambda)$ or $\mathbb{Q}H^n(\lambda)$, we shall denote by $r_p$ the function "distance to $p$" in the ambient space. We shall also use the notation $\partial_{r_p} = \mathrm{grad}\, r_p$.

**Lemma 5** ([11], [18]). *In $\mathcal{M}$, if* $\dim(\mathcal{M}) = n$, *then*

$$\nabla^2 r_p(X, X) \begin{cases} = 0 \text{ if } X = \partial_{r_p} \\ \geqq co_\lambda(r_p)|X|^2 \text{ if } \langle X, \partial_{r_p}\rangle = 0, \end{cases} \qquad \Delta r_p \leqq -(n-1)co_\lambda(r_p), \tag{2.1}$$

*and, if $Y(t)$ is a Jacobi field along the geodesic* $\exp_p t\left(\frac{\exp_p^{-1}(x)}{|\exp_p^{-1}(x)|}\right)$, *orthogonal to this geodesic, with* $Y(0) = 0$, *and* $|Y'(0)| = 1$, *then*

$$|Y(r_p(x))| \geqq s_\lambda(r_p(x)), \tag{2.2}$$

Moreover, if $f : \mathbb{R} \longrightarrow R$ is a $C^2$ function, and $f(r_p) := f \circ r_p$, then

$$\Delta f(r_p) = -f''(r_p) + f'(r_p)\, \Delta r_p. \tag{2.3}$$

**Lemma 6.** (a) *In $\mathcal{M}$,*

$$\mathrm{Volume}(\Omega) \leqq \int_M \varphi_\lambda(r_p)\, \langle N, \partial_{r_p}\rangle. \tag{2.4}$$

*In particular, if* $\lambda = 0$, (2.4) *has the form* $\mathrm{Volume}(\Omega) \leqq \frac{1}{n} \int_M r_p\, \langle N, \partial_{r_p}\rangle$.

(b) *In* $\mathbb{C}H^n(\lambda)$,

$$2\, n\, \mathrm{Volume}(\Omega) = \int_M ta_\lambda(r_p)\, \langle N, \partial_{r_p}\rangle. \tag{2.5}$$

(c) *In* $\mathbb{Q}H^n(\lambda)$,

$$\text{(2.6)} \qquad \text{Volume}(\Omega) = \int_M \left( \frac{s_\lambda(r_p)}{4nc_\lambda^3(r_p)} - \frac{\lambda s_\lambda^3(r_p)}{(4n+2)c_\lambda^3(r_p)} \right) \langle N, \partial_{r_p} \rangle.$$

Proof. The formula (2.5) was obtained in ([16]) (following the ideas in [17]). We shall prove (2.4). By (2.1) and (2.3), taking $f(t) = \int_0^t \varphi_\lambda(s)ds$,

$$\Delta \left( \int_0^{r_p} \varphi_\lambda(s)ds \right) \leqq -\varphi_\lambda{}'(r_p) - (n-1)co_\lambda(r_p)\varphi_\lambda(r_p) = -1$$

and, by Stokes theorem,

$$\text{Volume}(\Omega) \leqq -\int_\Omega \Delta \left( \int_0^{r_p} \varphi_\lambda(s)ds \right) = \int_M \left\langle \text{grad} \left( \int_0^{r_p} \varphi_\lambda(s)ds \right), N \right\rangle$$
$$\leqq \int_M \varphi_\lambda(r_p) \langle N, \partial_{r_p} \rangle.$$

The proof of (2.6) is similar, but using as $f(t)$ the function

$$\frac{1}{(4n+2)\lambda} \left( \frac{1}{4n\, c_\lambda^2(t)} - \ln c_\lambda(t) \right). \quad \square$$

**Lemma 7.** *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a* $C^2$*-function satisfying that* $f'(t) > 0$ *and* $f''(t) > 0$ *for every* $t > 0$*. Every compact hypersurface* $M$ *of* $\mathcal{M}$ (*we suppose that, if* $\lambda > 0$, $M$ *is contained in a geodesic ball of radius lower than* $\frac{\pi}{2\sqrt{\lambda}}$), $\mathbb{C}H^n(\lambda)$ *or* $\mathbb{Q}H^n(\lambda)$ *has a unique* $f$*-center of mass.*

*In particular, any compact hypersurface* $M$ *in* $\mathcal{M}$ *has a unique* $\int_0^t \varphi_\lambda(s)ds$*-center of mass, any compact hypersurface* $M$ *in* $\mathbb{C}H^n(\lambda)$ *has a unique* $\ln c_\lambda(t)$*-center of mass and any compact hypersurface* $M$ *in* $\mathbb{Q}H^n(\lambda)$ *has a unique* $\frac{1}{(4n+2)\lambda}(\frac{1}{4nc_\lambda^2(t)} - \ln c_\lambda(t))$*-center of mass.*

The proof of this lemma is essentially contained in [6], we have just to change $r^2$ by $f(r)$ and, having account that $\mathbb{C}H^n(\lambda)$ and $\mathbb{Q}H^n(\lambda)$ have their sectional curvature bounded from above by $\lambda < 0$, use the formula (2.1). Along this proof it is shown that the $f$-center of mass $p$ of $M$ is the unique critial point of the function $\mathcal{F}$ defined in Section 1. Then, it satisfies that, for every $\zeta \in T_p\mathcal{M}$(or $T_p\mathbb{C}H^n(\lambda)$, or $T_p\mathbb{Q}H^n(\lambda)$ respectively),

$$\text{(2.7)} \qquad 0 = \langle \text{grad}\, \mathcal{F}(p), \zeta \rangle = \int_M f' \langle \partial_{r_p}, \zeta \rangle$$

**3. The proof of the theorems.** We shall give the details of the proof of Theorem 3. The other theorems are proved by using similar arguments and computations, with the help of the corresponding parts of Lemmas 5, 6 and 7.

In order to prove Theorem 3, we begin working with a generic $f$ satisfying the hypothesis of Lemma 7. Only in the final step we shall substitute it for its expression given in the statement of the theorem. Let $x_i$ denote the normal coordinates of $\mathbb{C}H^n(\lambda)$ centered at the $f$-center of mass $p$ of $M$, and let $r := r_p$. Since $f$ satisfies the hypotheses of Lemma 7, it follows from (2.7), taking $\zeta = e_i$, a unit vector of an orthonormal basis of $T_p\mathbb{C}H^n(\lambda)$ used to define normal coordinates in $\mathbb{C}H^n(\lambda)$ centered at $p$, that the functions

$$f_i = \frac{g}{r}\, x_i,\ i = 1, \ldots, 2n,\ \text{where } g = f',\ \text{satisfy} \int\limits_M f_i = 0. \tag{3.1}$$

On the other hand, using some results in [4], Rivertz and Tomter ([20, Th.1 and Lemma 1]) have shown that the eigenfunctions of the first eigenvalue of the laplacian of a geodesic sphere $\partial B_R$ of radius $R$ in $\mathbb{C}H^n(\lambda)$, with center at $p$, are the simultaneous eigenfunctions of the vertical and horizontal laplacians of the euclidean sphere $S^{2n-1}$ of radius $s_\lambda(R)$ in $\mathbb{C}^n$. It is easy to check that the real coordinate functions in $\mathbb{C}^n$ of the points in $S^{2n-1}$ are simultaneous eigenfunctions of the vertical and horizontal laplacians. By identification of $\mathbb{C}^n$ with $T_p\mathbb{C}H^n(\lambda)$, these coordinate functions of $S^{2n-1}$ become the normal coordinate functions $x_i$ of $\partial B_R$ times the factor $1/s_\lambda(R)$. Then the normal coordinate functions $x_i$ are eigenfunctions corresponding to the first eigenvalue of $\partial B_R$. For this reason, and (3.1), we shall use these $f_i$ as test functions to apply Raileigh's principle

$$\lambda_1(M) \leqq \frac{\int\limits_M |\mathrm{grad}_M\, f_i|^2}{\int\limits_M f_i^2} \tag{3.2}$$

to bound $\lambda_1$. Let us compute the gradients (in $M$) of the functions $f_i$ in a point $x \in M$:

$$\mathrm{grad}_M\, f_i = x_i\, \frac{rg' - g}{r^2}\, \partial_r^\top + \frac{g}{r}\mathrm{grad}_M\, x_i, \tag{3.3}$$

if $\{e_1, \ldots, e_{2n-2}, JN\}$ is an orthonormal basis of $T_xM$,

$$|\mathrm{grad}_M\, f_i|^2 = \sum_{j=1}^{2n-2} \langle \mathrm{grad}_M\, f_i, e_j\rangle^2 + \langle \mathrm{grad}_M\, f_i, JN\rangle^2,$$

and, from (3.3)

$$\begin{aligned}\sum_{i=1}^{2n} \langle \mathrm{grad}_M\, f_i, JN\rangle^2 = {}& \frac{r^2 g'^2 + g^2 - 2rgg'}{r^2}\, \langle \partial_r, JN\rangle^2 \\ & + \frac{g^2}{r^2} \sum_{i=1}^{2n} \langle \mathrm{grad}_M\, x_i, JN\rangle^2 \\ & + \frac{2rgg' - 2g^2}{r^3}\, \langle \partial_r, JN\rangle \left\langle \sum_{i=1}^{2n} x_i \mathrm{grad}_M\, x_i, JN \right\rangle,\end{aligned}$$

but $\sum\limits_{i=1}^{2n} x_i \operatorname{grad}_M x_i = \frac{1}{2}\operatorname{grad}_M r^2 = r\partial_r^\top$, and

$$\sum_{i=1}^{2n}\langle \operatorname{grad}_M f_i, JN\rangle^2 = \frac{r^2 g'^2 - g^2}{r^2}\langle\partial_r, JN\rangle^2 + \frac{g^2}{r^2}\sum_{i=1}^{2n}\langle \operatorname{grad}_M x_i, JN\rangle^2. \tag{3.4}$$

In order to compute $\sum\limits_{i=1}^{2n}\langle \operatorname{grad}_M x_i, JN\rangle^2$, we decompose

$$JN = \mu_r\partial_r + \mu_J J\partial_r + \nu\xi, \tag{3.5}$$

where $\xi$ is orthogonal to $\partial_r$ and $J\partial_r$ and $|\xi| = 1$. Since $JN$ is tangent to $M$,

$$\langle \operatorname{grad}_M x_i, JN\rangle = \langle \operatorname{grad}_M x_i, JN\rangle = dx_i(JN). \tag{3.6}$$

Moreover,

$$dx_i(\partial_r) = dx_i\left(\frac{1}{r}\sum_{j=1}^{2n-2} x_j\frac{\partial}{\partial x_j}\right) = \frac{x_i}{r}, \tag{3.7}$$

$$dx_i(J\partial_r) = dx_i\left(\frac{1}{s_{4\lambda}(r)}s_{4\lambda}(r)J\partial_r\right) = \frac{r}{s_{4\lambda}(r)}(Ju)_i, \tag{3.8}$$

where $u$ is the unit vector in $T_p\mathbb{C}H^n(\lambda)$ tangent to the geodesic from $p$ to $x$, because $Y_J(r) := s_{4\lambda}(r)J\partial_r$ is the Jacobi field along this geodesic satisfying $Y_J{}'(0) = Ju$, and $x_i$ are normal coordinates. Analogously

$$dx_i(\xi) = dx_i\left(\frac{1}{s_\lambda(r)}s_\lambda(r)\xi\right) = \frac{r}{s_\lambda(r)}(\xi_p)_i, \tag{3.9}$$

where $\xi_p$ is the unit vector in $T_p\mathbb{C}H^n(\lambda)$ such that $\xi$ is the parallel transport of $\xi_p$ along the same geodesic.

From (3.5), (3.6), (3.7), (3.8) and (3.9),

$$\begin{aligned}\sum_{i=1}^{2n}\langle \operatorname{grad}_M x_i, JN\rangle^2 &= \mu_r^2\sum_{i=1}^{2n}\frac{x_i^2}{r^2} + \mu_J^2\frac{r^2}{s_{4\lambda}^2(r)}\sum_{i=1}^{2n}(Ju)_i^2 \\ &\quad + \nu^2\frac{r^2}{s_\lambda^2(r)}\sum_{i=1}^{2n}(\xi_p)_i^2 + 2\mu_r\mu_j\frac{1}{s_{4\lambda}}(r)\sum_{i=1}^{2n}x_i(Ju)_i \\ &\quad + 2\mu_r\frac{\nu}{s_\lambda(r)}\sum_{i=1}^{2n}x_i(\xi_p)_i + 2\mu_J\frac{\nu\, r^2}{s_\lambda(r)s_{4\lambda}(r)}\sum_{i=1}^{2n}(Ju)_i(\xi_p)_i \\ &= \mu_r^2 + \mu_J^2\frac{r^2}{s_{4\lambda}^2(r)} + \nu^2\frac{r^2}{s_\lambda^2(r)}\end{aligned}$$

$$= \langle JN, \partial_r\rangle^2 + \langle N, \partial_r\rangle^2 \frac{r^2}{s_{4\lambda}^2(r)} + (1 - \langle N, \partial_r\rangle^2$$

$$- \langle JN, \partial_r\rangle^2)\frac{r^2}{s_\lambda^2(r)} = \langle JN, \partial_r\rangle^2 + \langle N, \partial_r\rangle^2 \frac{r^2}{s_{4\lambda}^2(r)} + |\partial_r^J|^2 \frac{r^2}{s_\lambda^2(r)}, \tag{3.10}$$

where $\partial_r^J$ is the projection of $\partial_r$ onto the orthogonal to $N$ and $JN$. From (3.4) and (3.10), we obtain

$$\sum_{i=1}^{2n} \langle \operatorname{grad}_M f_i, JN\rangle^2 = g'^2(r)\, \langle \partial_r, JN\rangle^2 + \frac{g^2(r)}{s_{4\lambda}^2(r)} \langle \partial_r, N\rangle^2 + \frac{g^2(r)}{s_\lambda^2(r)} |\partial_r^J|^2. \tag{3.11}$$

A similar computation gives

$$\sum_{i=1}^{2n} \langle \operatorname{grad}_M x_i, e_j\rangle^2$$

$$= \langle \partial_r, e_j\rangle^2 + \langle e_j, J\partial_r\rangle^2 \frac{r^2}{s_{4\lambda}^2} + \ (1 - \langle e_j, \partial_r\rangle^2 - \langle e_j, J\partial_r\rangle^2)\frac{r^2}{s_\lambda^2}, \tag{3.12}$$

$$\sum_{j=1}^{2n-2} \sum_{i=1}^{2n} \langle \operatorname{grad}_M f_i, e_j\rangle^2 = g'^2 |\partial_r^J|^2 + \frac{g^2}{s_{4\lambda}^2} |\partial_r^J|^2 + 2\frac{g^2}{s_\lambda^2}(n - 1 - |\partial_r^J|^2). \tag{3.13}$$

From (3.13) and (3.11), having account that $\frac{1}{s_{4\lambda}^2} - \frac{1}{s_\lambda^2} = \frac{\lambda}{c_\lambda^2}$, we obtain

$$\sum_{i=1}^{2n} |\operatorname{grad}_M f_i|^2$$

$$= \left(g'^2 + \frac{\lambda}{c_\lambda^2} g^2\right) |\partial_r^J|^2 + g'^2\, \langle \partial_r, JN\rangle^2 + \ \frac{g^2}{s_{4\lambda}^2} \langle \partial_r, N\rangle^2 + (2n-2)\frac{g^2}{s_\lambda^2}. \tag{3.14}$$

On the other hand,

$$\sum_{i=1}^{2n} \int_M f_i^2 = \int_M \left( \sum_{i=1}^{2n} \frac{g^2}{r^2} x_i^2 \right) = \int_M g^2. \tag{3.15}$$

From (3.2), (3.14) and (3.15), it follows

$$\lambda_1 \int_M g^2 \leqq \int_M \left( \left( g'^2 + \frac{\lambda}{c_\lambda^2} g^2 \right) |\partial_r^J|^2 + g'^2\, \langle \partial_r, JN\rangle^2 \right.$$

$$\left. + \frac{g^2}{s_{4\lambda}^2} \langle \partial_r, N\rangle^2 + (2n-2)\frac{g^2}{s_\lambda^2.} \right). \tag{3.16}$$

In order to combine this inequality with (2.5), by Lemma 7, we choose $g(t) = ta_\lambda(t)$ (that is, $f(t) = \ln c_\lambda(t)$). Then

$$g'^2 = \frac{g^2}{s_{4\lambda}^2} = \frac{1}{c_\lambda^4} \quad \text{and} \quad g'^2|\partial_r^J|^2 + g'^2 \langle \partial_r, JN\rangle^2 + \frac{g^2}{s_{4\lambda}^2} \langle \partial_r, N\rangle^2 = \frac{1}{c_\lambda^4}.$$

By substitution in (3.16), we obtain

$$(3.17) \qquad \lambda_1 \int_M ta_\lambda^2 \leqq \int_M \left( \lambda \frac{s_\lambda^2}{c_\lambda^4} |\partial_r^J|^2 + \frac{1 + (2n-2)c_\lambda^2}{c_\lambda^4} \right) \leqq \int_M \left( \frac{1 + (2n-2)c_\lambda^2}{c_\lambda^4} \right),$$

since $\lambda < 0$. On the other hand, by Schwarz inequality and (2.5)

$$(3.18) \qquad \text{Volume}(M) \int_M ta_\lambda^2 \geqq \left( \int_M ta_\lambda \right)^2 \geqq 4\, n^2 \; (\text{Volume}(\Omega))^2 .$$

From (3.18) and (3.17) we obtain

$$\lambda_1 \leqq \frac{\text{Volume}(M)}{4\, n^2 \; (\text{Volume}(\Omega))^2} \int_M \left( \frac{1 + (2n-2)c_\lambda^2}{c_\lambda^4} \right),$$

with the equality if and only if the inequalities (3.18), (3.17) y (3.16) are equalities, which happens (for each one of the inequalities) if and only if $M$ is a geodesic sphere and $\Omega$ a geodesic ball in $\mathbb{C}H^n(\lambda)$. From this, the fact that the function $\frac{1+(2n-2)c_\lambda^2}{c_\lambda^4}$ is decreasing, and elementary computations using (1.2), Theorem 3 follows.

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