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Statistical analysis when dealing with astigmatism: assessment of different spherocylindrical notations

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Abstract Ophthalmic epidemiological studies frequently deal with ocular refractive errors, which are commonly expressed in the form sphere/cylinder x axis. However, this representation has been shown not to be the most suitable one for performing statistical analysis. Although alternative analytical and graphic methods to represent this kind of data have been developed, these formalisms have often gone unnoticed by researchers, despite their usefulness and versatility. Besides, there has been no discussion of how each of them fits in with a particular type of study.

In this paper, several mathematical representations of dioptric power are revisited in a comprehensive way. The aim is to encourage researchers in ophthalmology and optometry to use these formalisms in their epidemiological studies, thus profiting from their exactitude and simplicity. Consequently, the emphasis is not on complicated mathematical derivations but on how to use these representations. Their potential and suitability in different applications is analyzed in detail. In addition, some examples are presented to illustrate the mathematical methods considered.

Key words Astigmatism; refractive errors; dioptric power matrix; statistical analysis

Introduction The standard representation of dioptric power expressed in the form sphere/cylinder x axis is satisfactory for traditional clinical purposes but unsuited to mathematical and statistical processing, such as that needed in certain epidemiological studies. For example, when it is necessary to add or subtract spherocylinders to calculate the internal ocular astigmatism or to determine the axis misalignment on fitting soft contact lenses, the use of the standard notation is rather cumbersome since the units and the measuring system of each of the three parameters are different.[1] In certain cases, the use of the spherical equivalent as a single parameter overcomes this drawback. However, this is certainly not the best choice because part of the information is lost, since the same spherical equivalent can be obtained from different refractive errors. A consequence is the lack of consistency of the results.

As statistical analysis is an outstanding component of ophthalmic epidemiological studies, the introduction of modern scientific representations of dioptric power is necessary to take into account not only the spherical equivalent but the relative influence of each of the three parameters that define a refractive status. Besides, these representations provide an easy and unambiguous way to add or subtract spherocylinders[2] and even permit a graphic representation of results in an Euclidean space.

Several formalisms have been developed, starting from the matrix representation of dioptric power proposed by Long.[3] Some of them are extremely simple and very well adapted to represent and analyze statistical distributions involving spherocylindrical powers.[4–6] Besides, each has its own advantages and all of them constitute different alternatives to face a given problem. In this paper, their relative performance and advantages in some clinical statistical applications are discussed. Although our aim is to show how and when these methods should be used, avoiding heavy mathematical discussions, a brief description of the different methods is presented first for the sake of completeness.

The Long formalism One of the main disadvantages of representing the dioptric power in the traditional notation is its ambiguity. Actually, refractive errors can be expressed in three ways: the bicylindrical expression, the minus cylinder one or its transposition. This fact becomes a drawback when it is desired to collect data from several clinicians so as to perform an epidemiological study. The mathematical representation of dioptric power as a matrix in the way developed by Long[3] overcomes this shortcoming since it is invariant during transposition of the prescription. In this sense, the power of an astigmatic surface – or thin lens – can be represented by the power along its principal meridians. If they form an angle α with the coordinate system – in the standard notation S/C × α –, the dioptric power of the surface can be expressed as:[3]

$$F=\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}=\begin{bmatrix} S+C\sin^2\alpha & -C\sin\alpha\cos\alpha \\ -C\sin\alpha\cos\alpha & S+C\cos^2\alpha \end{bmatrix}. \tag{1}$$

Eq.(1) is the dioptric power matrix of a spherocylinder defined by S/C × α. It must be noted that this matrix is symmetric and its diagonal elements represent the power of the lens in the horizontal and vertical meridians. It can be proven[7] that the mathematical expression to return to the standard representation (S/C × α) can be obtained from the trace ($t=f_{11}+f_{22}$) and the determinant ($d=f_{11}f_{22}-f_{12}f_{21}$) of the dioptric power matrix as follows:

$$C=\pm\sqrt{t^2-4d};\quad S=(t-C)/2;\quad \tan\alpha=(S-f_{11})/f_{12}. \tag{2}$$

The choice of the plus or minus cylinder sign in Eq.(2) is arbitrary, so that the final solution will be in the form of a plus or minus cylinder transposition.

Example 1 The refractive error $+1.00 \times 180°; -3.00 \times 90°$ expressed in bicylindrical form can also be expressed as $+1.00; -4.00 \times 90°$ or $-3.00; +4.00 \times 180°$. By the use of Eq.(1), these three notations converge to a single one given by the matrix:

$$F = \begin{bmatrix} S + C\sin^2\alpha & -C\sin\alpha\cos\alpha \\ -C\sin\alpha\cos\alpha & S + C\cos^2\alpha \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}.$$

This feature demonstrtates that the matrix notation of dioptric power is very useful to analyze data and to carry out numerical statistical studies, since data processing is immediate starting from these matrices, i. e., no intermediate step is required to normalize the notation of the collected data. Another important advantage of this representation is that the power resulting from the addition – or subtraction – of two or more spherocylindrical powers can easily be obtained by adding or subtracting its corresponding matrices.

Example 2 The addition of two spherocylinders with power $R_1 = +1.00; -2.00 \times 70°$ and $R_2 = +3.00; -1.00 \times 85°$ is certainly not obvious by the use of this notation, but expressed in the form of matrices the result is simply obtained:

$$F = F_1 + F_2 = \begin{bmatrix} -0.766 & 0.643 \\ 0.643 & 0.766 \end{bmatrix} + \begin{bmatrix} 2.008 & 0.087 \\ 0.087 & 2.992 \end{bmatrix} = \begin{bmatrix} 1.242 & 0.730 \\ 0.730 & 3.758 \end{bmatrix}.$$

From this result, the conventional notation can be obtained by the use of Eq.(2): $+4.00; -3.00 \times 75°$.

These advantages make the Long formalism a suitable method to compute the kind of spherocylindrical data that require mathematical processing. It has proven to be a useful tool in applications that imply linear operations on spherocylindrical powers, such as to study the internal astigmatism in a population (see Example 3) or the astigmatism induced by any ocular surgery.[8] In addition, this method permits one to determine easily the axis misalignment on fitting soft contact lenses or to check the nominal power of these manufactured lenses.[9] For instance, if A is the patient's ocular refraction and B the refraction obtained over a mislocated trial lens, then the subtraction A–B provides the clinician with the degree of axis mislocation together with the power of the lens. As the mislocated lens can be in error not only in its axis, but also in its power, this method permits one to verify *in situ* the actual back vertex power of the manufactured contact lens with respect to its specified back vertex power. Long representation is also used to study prismatic effects; to determine the prismatic deviation that produces a certain decentriment or, on the other hand, to calculate the decentriment necessary to generate a certain deviation.[3] Questions of this kind, which usually require a tedious calculation, are reduced to a simple algebraic operation by use the of the matrix formalism.

Example 3 Let us consider a hypothetical simulation devoted to seeing how the matrix method works. Suppose that a study has been carried out in order to determine the internal astigmatism of a certain

A_R	F_R	A_C	F_C	$F_R–F_C$*	A_I
$-3.25 \times 55°$	$\begin{bmatrix} -2.18 & 1.53 \\ 1.53 & -1.07 \end{bmatrix}$	$-3.00 \times 50°$	$\begin{bmatrix} -1.76 & 1.48 \\ 1.48 & -1.24 \end{bmatrix}$	$\begin{bmatrix} -0.42 & 0.05 \\ 0.05 & 0.17 \end{bmatrix}$	$-0.60 \times 85°$
$-2.50 \times 70°$	$\begin{bmatrix} -2.21 & 0.80 \\ 0.80 & -0.29 \end{bmatrix}$	$-1.75 \times 65°$	$\begin{bmatrix} -1.44 & 0.67 \\ 0.67 & -0.31 \end{bmatrix}$	$\begin{bmatrix} -0.77 & 0.13 \\ 0.13 & 0.02 \end{bmatrix}$	$-0.83 \times 81°$
$-0.50 \times 35°$	$\begin{bmatrix} -0.16 & 0.23 \\ 0.23 & -0.34 \end{bmatrix}$	$-1.00 \times 15°$	$\begin{bmatrix} -0.07 & 0.25 \\ 0.25 & -0.93 \end{bmatrix}$	$\begin{bmatrix} -0.10 & -0.20 \\ -0.02 & 0.60 \end{bmatrix}$	$-0.70 \times 91°$
$-1.25 \times 90°$	$\begin{bmatrix} -1.25 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}$	$-0.25 \times 95°$	$\begin{bmatrix} -0.32 & -0.02 \\ -0.02 & 0.00 \end{bmatrix}$	$\begin{bmatrix} -1.00 & 0.02 \\ 0.02 & 0.00 \end{bmatrix}$	$-1.00 \times 89°$
$-2.75 \times 120°$	$\begin{bmatrix} -2.06 & -1.19 \\ -1.19 & -0.69 \end{bmatrix}$	$-2.50 \times 125°$	$\begin{bmatrix} -1.68 & -1.17 \\ -1.17 & -0.82 \end{bmatrix}$	$\begin{bmatrix} -0.39 & -0.02 \\ -0.02 & 0.14 \end{bmatrix}$	$-0.52 \times 92°$
$-1.00 \times 75°$	$\begin{bmatrix} -0.93 & 0.25 \\ 0.25 & -0.07 \end{bmatrix}$	$-0.50 \times 50°$	$\begin{bmatrix} -0.29 & 0.25 \\ 0.25 & -0.21 \end{bmatrix}$	$\begin{bmatrix} -0.64 & 0.00 \\ 0.00 & 0.14 \end{bmatrix}$	$-0.78 \times 90°$
$-3.75 \times 160°$	$\begin{bmatrix} -0.44 & -1.21 \\ -1.21 & -3.31 \end{bmatrix}$	$-4.00 \times 165°$	$\begin{bmatrix} -0.27 & -1.00 \\ -1.00 & -3.73 \end{bmatrix}$	$\begin{bmatrix} -0.17 & -0.21 \\ -0.21 & 0.42 \end{bmatrix}$	$-0.72 \times 107°$
$-0.25 \times 10°$	$\begin{bmatrix} -0.01 & 0.04 \\ 0.04 & -0.24 \end{bmatrix}$	$-0.75 \times 5°$	$\begin{bmatrix} -0.01 & 0.07 \\ 0.07 & -0.74 \end{bmatrix}$	$\begin{bmatrix} 0.00 & -0.02 \\ -0.02 & 0.50 \end{bmatrix}$	$-0.51 \times 93°$
$-1.75 \times 100°$	$\begin{bmatrix} -1.70 & -0.30 \\ -0.30 & -0.05 \end{bmatrix}$	$-1.00 \times 105°$	$\begin{bmatrix} -0.93 & -0.25 \\ -0.25 & -0.07 \end{bmatrix}$	$\begin{bmatrix} -0.76 & -0.05 \\ -0.05 & 0.01 \end{bmatrix}$	$-0.78 \times 94°$
$-1.25 \times 15°$	$\begin{bmatrix} -0.08 & 0.31 \\ 0.31 & -1.17 \end{bmatrix}$	$-1.75 \times 10°$	$\begin{bmatrix} -0.05 & 0.30 \\ 0.30 & -1.70 \end{bmatrix}$	$\begin{bmatrix} -0.03 & 0.01 \\ 0.01 & 0.53 \end{bmatrix}$	$-0.56 \times 89°$

*This column may not add up due to rounding.

TABLE 1. Simulation of a sample to obtain the distribution of internal astigmatism by the use of Long matrices. A_R is the refractive astigmatism, A_C the corneal astigmatism and A_I the internal astigmatism. F_R and F_C are the matrices corresponding to A_R and A_C, respectively.

population. If A_R is the refractive astigmatism and A_C the corneal astigmatism, then the internal astigmatism (A_I) will be obtained from the difference of their respective matrices as $F_R - F_C$. In Table 1 we show the values A_R and A_C together with their associated matrices. In the last column, the internal astigmatism is easily obtained by subtracting the matrices. For simplicity we have restricted the sample to ten values.

Taking into account that Eq.(1) and Eq.(2) are easily programmable in a low-cost computer, the Long notation is a good choice when the aim of research is, for example, a numerical analysis of the results. In spite of this, it has been rated by some authors as rather abstract and inaccessible to clinicians,[10,11] perhaps due to the lack of a graphic representation to support the usefulness of the calculations. However, the most important feature of the matrix representation is that it provides a basis for other powerful representations of dioptric power, such as the three-dimensional spaces treated in the next section.

Representation of the dioptric power in a three-dimensional Euclidean space A graphic or geometric representation of the results is often required when carrying out a statistical analysis of refractive data expressed as S/C × α.[12] Harris[4] proposed a three-dimensional Euclidean space to represent this kind of data. He considered the matrix F of Eq.(1) as the linear combination of three specific dioptric powers:[13]

$$F=(S+C\sin^2\alpha)\begin{bmatrix}1&0\\0&0\end{bmatrix}+(S+C\cos^2\alpha)\begin{bmatrix}0&0\\0&1\end{bmatrix}$$
$$+(-\sqrt{2}C\cos\alpha\sin\alpha)\begin{bmatrix}0&1/\sqrt{2}\\1/\sqrt{2}&0\end{bmatrix}. \tag{3}$$

Provided that the three matrices in Eq.(3) can be considered as an orthonormal base of a three-dimensional Euclidean space, the coordinates corresponding to any dioptric power in such a space are given by the coefficients of Eq.(3). So, a specific dioptric power is defined by a single point in this space and can be determined by a vector h with coordinates:

$$h_1=S+C\sin^2\alpha \quad h_2=-\sqrt{2}C\cos\alpha\sin\alpha \quad h_3=S+C\cos^2\alpha \tag{4}$$

This vector is a very useful magnitude to quantify dioptric powers (see Figure 1). Comparison of Eq.(1) with Eq.(4) yields: $f_{11}=h_1, f_{21}=f_{12}=h_2/\sqrt{2}$ and $f_{22}=h_3$. Therefore, the standard notation will be reobtained by use of the same expression as with the Long formalism (Eq.(2)):

$$C=\pm\sqrt{t^2-4d};\quad S=(t-C)/2;\quad \tan\alpha=\frac{S-h_1}{h_2/\sqrt{2}}, \tag{5}$$

where $t=h_1+h_3$ and $d=h_1h_3-(h_2/\sqrt{2})^2$.

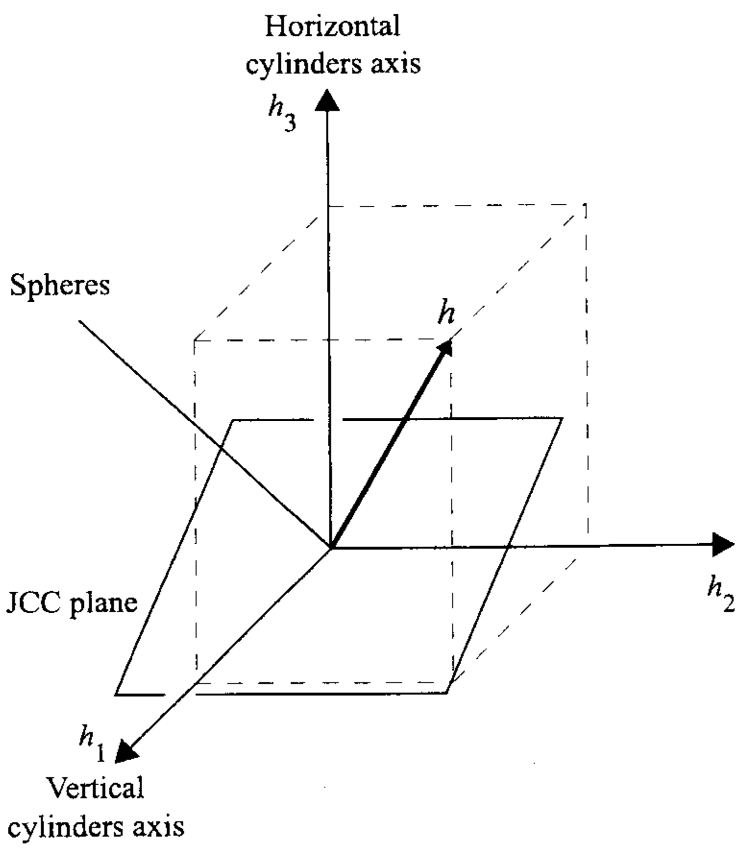

Fig. 1. Three-dimensional space obtained according to the Harris notation. The dioptric power is defined by the vector *h*.

Another useful orthogonal base of the three-dimensional dioptric power space can also be derived easily from Eq.(1). Expressing this last equation as:

$$F=(S+C\sin^2\alpha)\begin{bmatrix}1&0\\0&0\end{bmatrix}+(S+C\cos^2\alpha)\begin{bmatrix}0&0\\0&1\end{bmatrix}+(-C\cos\alpha\sin\alpha)\begin{bmatrix}0&1\\1&0\end{bmatrix}, \tag{6}$$

and by the use of trigonometric identities, we obtain:

$$F=\left(-\frac{C}{2}\cos 2\alpha\right)\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\left(-\frac{C}{2}\sin 2\alpha\right)\begin{bmatrix}0&1\\1&0\end{bmatrix}+\left(S+\frac{C}{2}\right)\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{7}$$

Therefore, the new base defined by the matrices in Eq.(7) constitute an alternative coordinate system for the dioptric power space, where a given power is fully specified by use of the coordinates:

$$C_0=-\frac{C}{2}\cos 2\alpha;\quad C_{45}=-\frac{C}{2}\sin 2\alpha;\quad M=S+\frac{C}{2}. \tag{8}$$

Figure 2 represents a typical power vector in this coordinate system. This base was first proposed by Deal and Toop[5] and later by other authors[15,16] using different approaches. Besides, various researchers had previously used the coefficients of Eq.(8), although they had not recognized them as the coordinate base of a three-dimensional space.[17–23] The scalar magnitude that characterizes a spherocylindrical power is now the modulus of the vector d that defines a dioptric power

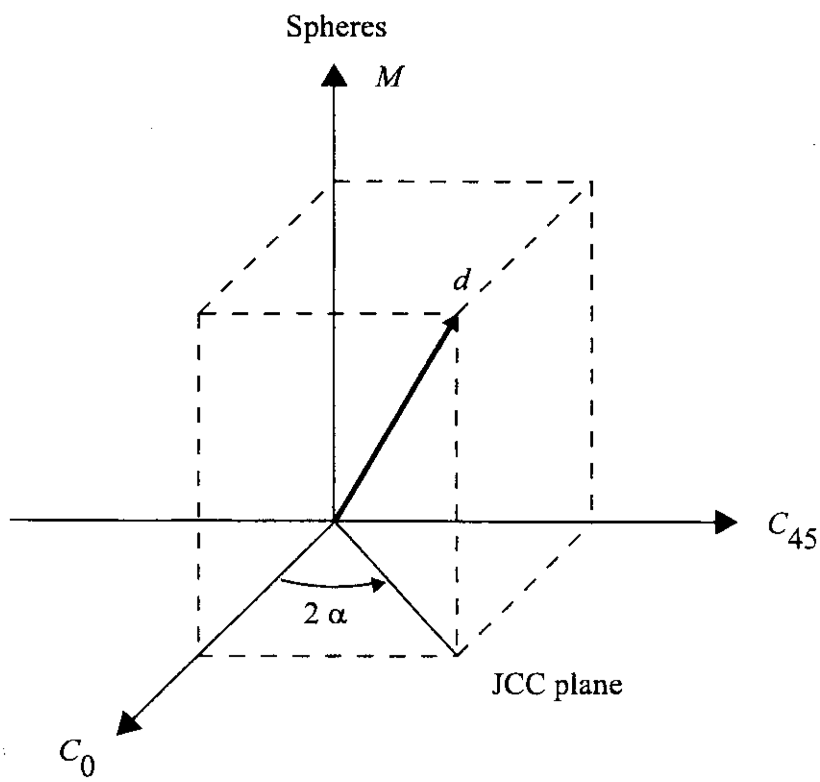

Fig. 2. Three-dimensional space obtained according to the Deal and Toop notation. The dioptric power is defined by the vector *d*.

in this coordinate system. Besides, according to Figure 2, it is obvious that we can return from the values C_0, C_{45} and M to the standard notation as:

$$C = -2\sqrt{C_0^2 + C_{45}^2} \quad S = M - \frac{C}{2} \quad \alpha = \frac{1}{2}\arctan\frac{C_{45}}{C_0} \tag{9}$$

Whenever C_0 is negative, then 90° must be added to α.

Both three-dimensional bases allow a graphic representation. In case of statistical studies, this kind of representation may be used to display, for instance, confidence regions about a mean power. Satisfying the requirements of statistics, in these three-dimensional representations distances are independent of the meridian taken as reference when the cylinder axis is determined. Therefore, the shape of confidence regions remains invariant, or preserved, during changes of the meridian chosen as reference.[24] Besides, since a single point in any of the three-dimensional spaces represents a certain spherocylindrical power, the superposition of different powers is consequently simply the addition of the vectors that define them.

In the three-dimensional space according to Harris notation, the axes h_1 and h_3 correspond to cylinders whose axes are vertical and horizontal, respectively, while the axis h_2 contains the Jackson cross-cylinders (JCC) with axes at 45° and 135°. The spherical powers lie on an axis perpendicular to h_2 (see Figure 1). A singular plane is the JCC plane. As its spherical equivalent is zero, this plane goes through the origin and, of course, contains the axis h_2.

On the other hand, the coordinate system of Figure 2 is perhaps more intuitive and easier to understand. Purely spherical powers appear along the axis M, while the Jackson cross-cylinders are represented in the plane $M = 0$ defined by the axes C_0 and C_{45}. Therefore, as any spherocylindrical power can be expressed as the combination of a pure sphere – the spherical equivalent – and a JCC lens, it admits a straightforward representation in this coordinate system. It must be pointed out that, independently of the coordinate system selected, the spheres form a line, the JCC a plane and the pure cylinders a cone.

Example 4 Let us perform the analysis of the data considered in *Example 3* by the use of these vector representations. From Eq.(4), we first obtain the vectors h_R, h_C and h_I, which correspond to A_R, A_C, and A_I (see Table 2). When the sample is plotted in Harris three-dimensional space (see Figure 3) it can be seen how the tips of the vectors corresponding to each item of information are distributed, within a certain range, along the axis h_1. Since h_1 contains the cylinders whose axes are vertical, from the graphic representation of the results it can be confirmed that the orientation of the internal astigmatism is about 90° and also that its power is around 0.50–0.75 D. It seems clear that, for the epidemiological study proposed in this example, the use of Harris coordinates of the power space is a very good choice because the axis h_1 itself contains the kind of dioptric powers we are working with (pure cylinders at 90°). This fact provides an easy interpretation of the results from the plot.

A_R	h_R	A_C	h_C	A_I	h_I	$\|h_I\|$
−3.25 × 55°	(−2.18, 2.16, −1.07)	−3.00 × 50°	(−1.76, 2.09, −1.24)	−0.60 × 85°	(−0.60, 0.07, 0.00)	0.60
−2.50 × 70°	(−2.21, 1.14, −0.29)	−1.75 × 65°	(−1.44, 0.95, −0.31)	−0.83 × 81°	(−0.81, 0.18, −0.02)	0.83
−0.50 × 35°	(−0.16, 0.33, −0.34)	−1.00 × 15°	(−0.07, 0.35, −0.93)	−0.70 × 91°	(−0.70, −0.02, 0.00)	0.70
−1.25 × 90°	(−1.25, 0.00, 0.00)	−0.25 × 95°	(−0.25, −0.03, 0.00)	−1.00 × 89°	(−1.00, 0.02, 0.00)	1.00
−2.75 × 120°	(−2.06, −1.68, −0.69)	−2.50 × 125°	(−1.68, −1.66, −0.82)	−0.52 × 92°	(−0.52, −0.03, 0.00)	0.52
−1.00 × 75°	(−0.93, 0.35, −0.07)	−0.50 × 50°	(−0.29, 0.34, −0.21)	−0.78 × 90°	(−0.78, 0.00, 0.00)	0.78
−3.75 × 160°	(−0.44, −1.70, −3.31)	−4.00 × 165°	(−0.26, −1.41, −3.73)	−0.72 × 107°	(−0.66, −0.28, −0.06)	0.72
−0.25 × 10°	(−0.01, 0.06, −0.24)	−0.75 × 5°	(−0.01, 0.09, −0.74)	−0.51 × 93°	(−0.51, −0.04, 0.00)	0.51
−1.75 × 100°	(−1.70, −0.42, −0.05)	−1.00 × 105°	(−0.93, −0.35, −0.07)	−0.78 × 94°	(−0.78, −0.08, 0.00)	0.78
−1.25 × 15°	(−0.08, 0.44, −1.17)	−1.75 × 10°	(−0.05, 0.42, −1.70)	−0.56 × 89°	(−0.56, 0.01, 0.00)	0.56

TABLE 2. Same sample as in Table I but the analysis is performed by the use of the Harris formalism. h_R, h_C and h_I are Harris vectors corresponding to A_R, A_C and A_I, respectively.

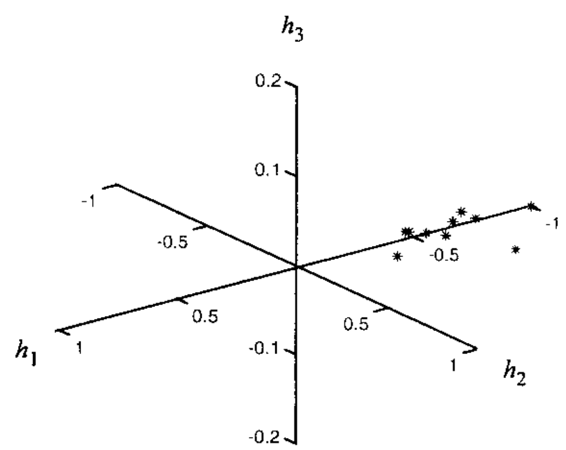

Fig. 3. Representation of the distribution of internal astigmatism in the Harris reference system from the sample in Table II.

On the other hand, when the data in *Example 3* are represented by the use of Deal and Toop notation, a set of different values results (see Table 3). As happens with Harris notation, the sample distribution can be analyzed from a single parameter, the modulus of d_I, which is represented in Figure 4 for the same sample. In this representation, as we plot pure cylinders with axes close to 90°, they lie along a line that forms an angle of 45° with the C_0 and M axis. Although, in this example, the Deal and Toop coordinate system seems to offer no advantages with respect to the Harris notation, there are other kinds of studies in which it does. In fact, the modulus of the vector d has an interesting feature, since all the vectors with the same modulus $|d|$ produce the same defocus that a spherical power of $\pm|d|$ D would produce and therefore a visual acuity can be associated with them.[25] For instance, the defocus produced by a single sphere of +0.50 D and a spherocylindrical lens of +0.50; −1.00 × 60° are equivalent. Then, the visual acuity associated with both refractions is the same. It can be verified by the use of Eq.(8) that the modulus $|d| = \sqrt{C_0^2 + C_{45}^2 + M^2}$ for both lenses is precisely 0.50. Thus, this fact permits us to estimate how visual acuity changes depend on the defocus from the modulus of d_I.

A_R	d_R	A_C	d_C	A_I	d_I	$\lvert d_I \rvert$
−3.25 × 55°	(−0.56, 1.53, −1.63)	−3.00 × 50°	(−0.26, 1.48, −1.50)	−0.60 × 80°	(−0.30, 0.05, −0.30)	0.42
−2.50 × 70°	(−0.96, 0.80, −1.25)	−1.75 × 65°	(−0.56, 0.67, −0.88)	−0.83 × 81°	(−0.39, 0.13, −0.42)	0.59
−0.50 × 35°	(0.09, 0.23, −0.25)	−1.00 × 15°	(0.43, 0.25, −0.50)	−0.70 × 91°	(−0.35, −0.01, −0.35)	0.50
−1.25 × 90°	(−0.63, 0.00, −0.63)	−0.25 × 95°	(0.13, −0.02, −0.13)	−1.00 × 89°	(−0.50, 0.02, −0.50)	0.71
−2.75 × 120°	(−0.69, −1.19, −1.38)	−2.50 × 125°	(−0.43, −1.17, −1.25)	−0.52 × 92°	(−0.26, −0.02, −0.26)	0.37
−1.00 × 75°	(−0.43, 0.25, −0.50)	−0.50 × 50°	(−0.04, 0.25, −0.25)	−0.78 × 90°	(−0.40, 0.00, −0.40)	0.55
−3.75 × 160°	(1.44, −1.21, −1.88)	−4.00 × 165°	(1.73, −1.00, −2.00)	−0.72 × 107°	(−0.30, −0.2, −0.36)	0.51
−0.25 × 10°	(0.12, 0.04, −0.13)	−0.75 × 5°	(0.37, 0.07, −0.38)	−0.51 × 93°	(−0.25, −0.03, −0.26)	0.36
−1.75 × 100°	(−0.82, −0.30, −0.88)	−1.00 × 105°	(−0.43, −0.25, −0.50)	−0.78 × 94°	(−0.39, −0.05, −0.40)	0.55
−1.25 × 15°	(0.54, 0.31, −0.63)	−1.75 × 10°	(0.82, 0.30, −0.88)	−0.56 × 89°	(−0.28, −0.01, −0.28)	0.40

TABLE 3. Same sample as in Table I but the analysis is performed by the use of the Deal and Toop formalism. d_R, d_C and d_I are the vectors corresponding to A_R, A_C and A_I, respectively.

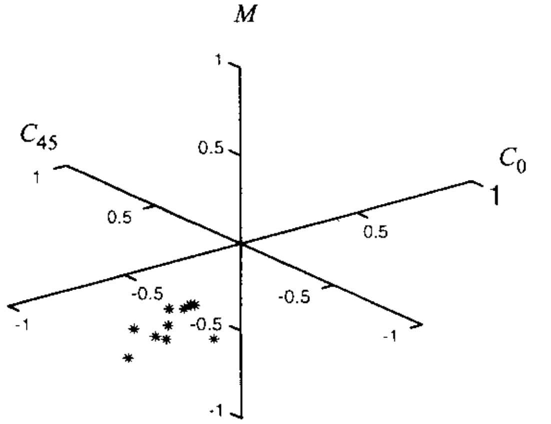

Fig. 4. Representation of the distribution of internal astigmatism in the Deal and Toop reference system from the sample in Table III. Note that the axes C_0 and C_{45} have been rotated.

The choice of a particular coordinate system depends on the specific application. For instance, if it is desired to plot only astigmatic values, then Harris notation provides the best interpretation of the results because the h_1 and h_3 coordinates correspond to the vertical and horizontal axes, respectively. On the other hand, due to its relationship with visual acuity, Deal and Toop notation is recommended to analyze the basis of certain refractive optometric techniques, such as the principle of operation of the JCC on refining both the axis and the power of the correcting cylinder during subjective refraction.[26] Both representations have found important applications recently, such as the analysis of the dynamic nature of the refractive ocular status determined by means of an autorefractometer,[27] the comparison of the refraction obtained with an autorefractometer and by the subjective method,[28] and to quantify differences between two values of dioptric power.[29] Another important feature is that these notations have also been used to visualize the time course of changes in refraction. In this way, changes in refraction over time are described by the trajectory of the tip of the moving vector in the power space.[30] Moreover, this trajectory can provide insights into a variety of clinical phenomena, such as the temporal evolution of the refractive status of the eye and its changes under several circumstances, such as after refractive surgery – quantifying residual refractive errors –, during ortho-keratology, during the progression of myopia, or in the course development of a disease that may affect the refractive status of the eye.

Discussion We can conclude that, on the one hand, Long's formalism fits quite well when the only purpose is to make a numerical analysis of the results. On the other hand, if a graphic representation is required, the use of three-dimensional power spaces is the best option. Depending on the kind of spherocylinders the researcher wishes to plot, different coordinate systems can be selected. In particular, if the results must be interpreted to analyze changes in visual acuity, it is preferable to choose the Deal and Toop representation. The use of the notations we have dealt with in this paper does not require complicated mathematics. Researchers need only change from the traditional notation to the most appropriate one by the use of Eq.(3), Eq.(6) or Eq.(10) and then statistical analysis can be applied as usual.

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