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Immersions of compact riemannian manifolds into a ball of a complex space form*

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1. Introduction

There are some classical theorems on non-immersibility of compact riemannian manifolds with sectional curvature bounded from above given by Tompkins, O'Neill, Chern, Kuiper and Moore (see [3], pages 221-226). More recently, attention has been paid to the case of immersions into a geodesic ball of a simply connected space form, and some conditions of non-immersibility in such a ball have been proved. In particular, estimates for the mean curvature of a complete immersion into a geodesic ball have been obtained by Jorge and Xavier [11] and a corresponding rigidity theorem for compact hypersurfaces has been proved by Markvorsen [14]. In this paper we give the Kähler analogs of the theorems of Jorge and Xavier (only for the compact case) and Markvorsen, and get some other new results for the Kähler case that have no Riemannian analog.

In order to state our results we shall introduce some notation and terminology.

Given a real number λ, let us consider the functions

$$s_\lambda(t) = \begin{cases} \dfrac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} \text{ if } \lambda > 0 \\ t \text{ if } \lambda = 0 \\ \dfrac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} \text{ if } \lambda < 0 \end{cases}, \qquad c_\lambda(t) = \begin{cases} \cos(\sqrt{\lambda}t) \text{ if } \lambda > 0 \\ 1 \text{ if } \lambda = 0 \\ \cosh(\sqrt{|\lambda|}t) \text{ if } \lambda < 0 \end{cases},$$

$$co_\lambda(t) = \frac{c_\lambda(t)}{s_\lambda(t)}.$$

These functions satisfy the following computation rules:

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$$s_\lambda{}' = c_\lambda, \quad c_\lambda{}' = -\lambda s_\lambda, \quad c_\lambda^2 + \lambda s_\lambda^2 = 1, \quad s_{4\lambda} = s_\lambda c_\lambda, \quad c_{4\lambda} = c_\lambda^2 - \lambda s_\lambda^2.$$

Let $\overline{M}$ be a Kähler manifold of real dimension $2n$, with Riemannian metric $\langle\ ,\ \rangle$, and almost complex structure J, and let p be a point of $\overline{M}$. Let M be a compact Riemannian manifold of dimension m and let $\psi : M \longrightarrow \overline{M}$ be an isometric immersion. Let $r : \overline{M} \longrightarrow \mathbb{R}$ be the distance to p in $\overline{M}$, and denote also by r the composition $r \circ \psi$. Let us denote by ∂_r the gradient of r in $\overline{M}$, and by $\partial_r^\top$ the vector field on M defined by $\partial_r^\top(q) = \psi_*^{-1}(\wp_q(\partial_r(\psi(q))))$ for every $q \in M$, where $\wp_q$ denotes the orthogonal projection $\wp_q : T_{\psi(q)}\overline{M} \longrightarrow \psi_* T_q M$.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is any function, $f(r)$ will denote the composition $f \circ r : M \longrightarrow \mathbb{R}$.

Let us denote by $cut(p)$ the set of cut points of p in $\overline{M}$. Let $R(r)$ be the $(1,1)$-tensor field on $\overline{M} - cut(p)$ defined by

$$R(r)A = R(\partial_r, A)\partial_r, \text{ for all vector fields } A \text{ on } \overline{M} - cut(p),$$

where R is the curvature tensor of $\overline{M}$, and let $R_\lambda(r)$ be the corresponding operator on the complex space form $\mathbb{K}^n(\lambda)$ of constant holomorphic sectional curvature 4λ ($\mathbb{K}^n(\lambda) = \mathbb{C}P^n(\lambda)$ if $\lambda > 0$ and $\mathbb{K}^n(\lambda) = \mathbb{C}H^n(\lambda)$ if $\lambda < 0$). Let us denote also by $R(r)$, $R_\lambda(r)$ the quadratic forms associated to these operators and the metrics on $\overline{M}$ or $\mathbb{K}^n(\lambda)$.

Let $c(p)$ be the cut distance from p in $\overline{M}$. For every $t < \min\{c(p), \frac{\pi}{2\sqrt{\lambda}}\}$, if $\lambda > 0$, and every $t < c(p)$, if $\lambda \leq 0$, we shall define the "transplanted" operator $\overline{R}_\lambda(r)$ of $R_\lambda(r)$ in $\overline{M}$ in the following way: let $\phi : T_{p'}\mathbb{K}^n(\lambda) \longrightarrow T_p\overline{M}$ be a holomorphic isometry, then

$$\overline{R}_\lambda(r)A(q) = \overline{\tau}_t \circ \phi \circ \tau_t^{-1} \circ R_\lambda(r) \circ \tau_t \circ \phi^{-1}(\overline{\tau}_t^{-1}A), \qquad t = \text{ distance } (p,q) \text{ in } \overline{M},$$

where $\overline{\tau}_t$ is the parallel transport along the minimizing geodesic γ of $\overline{M}$ from p to q and τ_t is the parallel transport along the minimizing geodesic $\exp_{p'} s\phi^{-1}(\gamma'(0))$ of $\mathbb{K}^n(\lambda)$ from p' to $\exp_{p'} t\phi^{-1}(\gamma'(0))$.

Given two quadratic forms, A and B, we shall say that $A \geq B$ if $A - B$ is positive semidefinite.

In the following, $\mathcal{N}M$ will denote the normal bundle on M associated to the immersion ψ, and we shall denote by J both the almost complex structure of the complex manifold $\overline{M}$ and the one induced by this on the vector bundle $\mathcal{N}M \oplus TM$.

From now on, ρ will denote a positive real number satisfying $\rho < \min\{c(p), \frac{\pi}{2\sqrt{\lambda}}\}$ if $\lambda > 0$ and $\rho < c(p)$ if $\lambda \leq 0$.

We shall denote by $B_\rho^{\lambda,n}$ a geodesic ball of $\mathbb{K}^n(\lambda)$ of radius ρ and by $\partial B_\rho^{\lambda,n}$ the corresponding geodesic sphere.

Our first theorem is the following Kähler version of Jorge and Xavier (compact case) and Markvorsen theorems. Let H denote the mean curvature associated to the immersion ψ, $B_\rho(p)$ the geodesic ball of radius ρ and centre p in $\overline{M}$, and $\partial B_\rho(p)$ the geodesic sphere; then we have

Theorem 1.1. *Let us suppose that* $R(r) \leq \overline{R}_\lambda(r)$ *in* $B_\rho(p)$, $\psi(M) \subset B_\rho(p)$, *and*

$$m|H| \leq (m-1)co_\lambda(\rho) + co_{4\lambda}(\rho).$$

When $\lambda \leq 0$, let us assume, also, that $J\mathcal{N}M \subset TM$. Then $\psi(M) \subset \partial B_\rho(p)$, $mH = -\{(m-1)co_\lambda(\rho) + co_{4\lambda}(\rho)\}\partial_r$, and $J\partial_r$ is tangent to $\psi(M)$ at each point.

If $m = 2n-1$, then $B_\rho(p)$ is isometric to a geodesic ball of radius ρ in $\mathbb{K}^n(\lambda)$, M is isometric to $\partial B_\rho^{\lambda,n}$ and ψ is an embedding.

Let us observe that the upper bound on $|H|$ implies that, for $\lambda > 0$, ρ must be not greater than the first zero of $m\cos^2(\sqrt{\lambda}t) - \sin^2(\sqrt{\lambda}t)$.

We remark, also, that the condition $J\mathcal{N}M \subset TM$ for $\lambda < 0$ is automatically fulfilled when $m = 2n-1$.

When $\lambda < 0$ and ρ tends to infinity, we have, as a consequence of Theorem 1.1, that there is no isometric immersion of a compact riemannian manifold in $\mathbb{C}H^n(\lambda)$ satisfying $|H| < (m+1)\sqrt{|\lambda|}$ and $J\mathcal{N}M \subset TM$.

Some of the new results that appear in the Kähler case are described in the following corollaries. Here $\Pi : \partial B_\rho^{\lambda,n} \longrightarrow \mathbb{C}P^{n-1}(1/s_\lambda^2(\rho))$ denotes the Hopf fibration on the geodesic sphere $\partial B_\rho^{\lambda,n}$ (see Sect. 4 for a description of it).

Corollary 1.2. *Let M be of dimension $2n-2$. Let us suppose that $\overline{M} = \mathbb{K}^n(\lambda)$, $\psi(M) \subset B_\rho^{\lambda,n}$ and $(2n-2)|H| \leq (2n-3)co_\lambda(\rho) + co_{4\lambda}(\rho)$. When $\lambda \leq 0$, we assume also that $J\mathcal{N}M \subset TM$. Then there is a compact riemannian manifold G of dimension $2n-3$, a riemannian submersion $\pi : M \longrightarrow G$, and a minimal immersion $\varphi : G \longrightarrow \mathbb{C}P^{n-1}(1/s_\lambda^2(\rho))$ such that $\Pi \circ \psi = \varphi \circ \pi$. Moreover, $\psi(M) = \Pi^{-1}(\varphi(G))$.*

Corollary 1.3. *Let M and ψ be as in Corollary 1.2, but with $n = 2$. Then $\psi : M \longrightarrow \mathbb{K}^2(\lambda)$ is an isometric immersion of a flat torus in $\mathbb{K}^2(\lambda)$, such that $\psi(M)$ is an embedded torus and also a tubular hypersurface of radius $(\pi/4)s_\lambda(\rho)$ around an integral curve of $J\partial_r$ in $\partial B_\rho^{\lambda,2}$.*

A look at the proof of Corollary 1.3, that we shall see in Sect. 4, shows that the only minimal embedded compact surfaces of $\partial B_\rho^{\lambda,n}$ containing the integral curves of $J\partial_r$ (or, equivalently, being totally real in $\mathbb{K}^2(\lambda)$) are the tubes mentioned therein. This gives some support to the "conjecture" that there are only a finite number of these embeddings (cfr. [1]).

We shall prove Theorem 1.1 in Sect. 3 and its corollaries in Sect. 4. Section 2 is devoted to some preliminary computations.

The bound $(m-1)co_\lambda(\rho) + co_{4\lambda}(\rho)$ is the lowest possible sum of principal curvatures of $\partial B_\rho^{\lambda,n}$ when $\lambda > 0$ and the biggest one for $\lambda < 0$. This explains that, for $\lambda < 0$, we have to add the condition $J\mathcal{N}M \subset TM$ to get rigidity, and poses the problem of existence of immersions without this condition and lower bounds on the mean curvature. Then, in Sect. 5 we give a theorem on immersions in the complex hyperbolic space (Theorem 5.1) without the above condition but with a lower bound on $|H|$ (the minimum possible sum of principal curvatures of $\partial B_\rho^{\lambda,n}$ when $\lambda < 0$).

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2. A formula for the laplacian (in M) of a function which is radial in $\overline{M}$

From now on, since all the computations are local, we shall denote by the same symbol local sections of $TM \oplus \mathcal{N}M$, the corresponding local vector fields on $\overline{M}$ along ψ, and their local extensions to local vector fields on $\overline{M}$.

We denote by $S(r)$ the $(1,1)$-tensor field on $\overline{M} - cut(p)$ defined by

$$S(r)(A) = -\overline{\nabla}_A \partial_r \text{ for every } A \text{ tangent to } \overline{M} - cut(p),$$

The following formula is well known (cfr. [9])

$$\overline{\nabla}^2 r(A, B) = -\langle S(r)A, B\rangle \text{ for every } A, B \text{ tangent to } \overline{M} - cut(p), \tag{2.1}$$

which implies

$$\overline{\Delta} r = \operatorname{tr} S(r). \tag{2.2}$$

Let us observe that $S(r)\partial_r = 0$ and $S(r)$ restricted to the tangent space to the geodesic sphere $\partial B_\rho(p)$ of $\overline{M}$ of centre p and radius ρ is the Weingarten map of this sphere, and $\operatorname{tr} S(r)$ is $(2n-1)$ times the mean curvature of this sphere.

On the other hand, an easy computation (see [10]) shows that

$$\nabla^2 r(X, Y) = \overline{\nabla}^2 r(X, Y) + \langle \alpha(X, Y), \partial_r\rangle \tag{2.3}$$

for every X, Y tangent to M, where α denotes the second fundamental form of the immersion $\psi : M \longrightarrow \overline{M}$.

If $\{e_i\}_{i=1}^m$ is a local orthonormal frame of vector fields tangent to M and H denotes the mean curvature of M, from (2.1) and (2.3), it follows that

$$\Delta r = \sum_{i=1}^m \langle S(r)e_i, e_i\rangle - m\langle H, \partial_r\rangle. \tag{2.4}$$

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is any C^2 function, we consider the function $f(r) = f \circ r : M \longrightarrow \mathbb{R}$. Then, it follows from (2.4) that

$$\Delta f(r) = -f''(r)|\partial_r^\top|^2 + f'(r)\left\{\sum_{i=1}^m \langle S(r)e_i, e_i\rangle - m\langle H, \partial_r\rangle\right\}. \tag{2.5}$$

When $\overline{M} = \mathbb{K}^n(\lambda)$, we denote by $S_\lambda(r)$ the operator $S(r)$. Let us denote also by $S(r)$, $S_\lambda(r)$ the quadratic forms associated to these operators and the metrics on $\overline{M}$ and $\mathbb{K}^n(\lambda)$ respectively. The transplanted operator $\overline{S}_\lambda(r)$ on $\overline{M}$ is defined from $S_\lambda(r)$ in the same way $\overline{R}_\lambda(r)$ is defined from $R_\lambda(r)$. The following Lemma is essentially well known

Lemma 2.1. *Let $\overline{M}$ be a Kähler manifold of real dimension $2n$, such that, for some $p \in \overline{M}$, $R(r) \le \overline{R}_\lambda(r)$ in $B_t(p)$ for some $t \le c(p)$ if $\lambda \le 0$, and $t \le \min\{c(p), \frac{\pi}{\sqrt{\lambda}}\}$, if $\lambda > 0$; then*

$$S(r) \le \overline{S}_\lambda(r) \text{ for every } q \in B_t(p). \tag{2.1.1}$$

Moreover, if the equality in (2.1.1) holds in $\partial B_\rho(p)$, for some $\rho < t$, then there exists a holomorphic isometry between $B_\rho(p)$ and $B_\rho^{\lambda,n}$ which takes $\partial B_\rho(p)$ onto $\partial B_\rho^{\lambda,n}$.

Proof . The inequality (2.1.1) is proved in [9], [16] and [4], while the characterization of the equality can be seen in [12] or [5].

On the other hand, the expression for $\overline{S}_\lambda(r)$ is the following (cfr. [6], page 138):

$$\overline{S}_\lambda(r)\partial_r = 0, \quad \overline{S}_\lambda(r)J\partial_r = -co_{4\lambda}(r)J\partial_r, \text{ and } \overline{S}_\lambda(r)X = -co_\lambda(r)X \tag{2.6}$$

for every X orthogonal to ∂_r and $J\partial_r$.

Then, under the hypotheses of Theorem 1.1, using (2.5) and (2.6), and the inequality of Lemma 2.1, we have, if $f' \ge 0$,

$$\begin{aligned}
\Delta f(r) &= -f''(r)|\partial_r^\top|^2 + f'(r)\left\{\sum_{i=1}^m \langle S(r)e_i, e_i\rangle - m\langle \partial_r, H\rangle\right\}\\
&\le -f''(r)|\partial_r^\top|^2 + f'(r)\left\{\sum_{i=1}^m \langle \overline{S}_\lambda(r)(e_i - \langle e_i, \partial_r\rangle\partial_r - \langle e_i, J\partial_r\rangle J\partial_r), e_i\rangle\right\}\\
&\qquad + f'(r)\left\{\sum_{i=1}^m \langle \overline{S}_\lambda(r)(\langle e_i, J\partial_r\rangle J\partial_r), e_i\rangle - m\langle \partial_r, H\rangle\right\}\\
&= -f''(r)|\partial_r^\top|^2 + f'(r)\left\{\sum_{i=1}^m -co_\lambda(r)\langle e_i - \langle e_i, \partial_r\rangle\partial_r - \langle e_i, J\partial_r\rangle J\partial_r, e_i\rangle\right\}\\
&\qquad + f'(r)\left\{\sum_{i=1}^m \{-co_{4\lambda}(r)|\langle e_i, J\partial_r\rangle|^2\} - m\langle \partial_r, H\rangle\right\}\\
&= -f''(r)|\partial_r^\top|^2 + f'(r)\,co_\lambda(r)|\partial_r^\top|^2\\
&\qquad + f'(r)\{(-co_{4\lambda}(r) + co_\lambda(r))\,|(J\partial_r)^\top|^2 - m\,co_\lambda(r) - m\,\langle \partial_r, H\rangle\}\\
&= (-f''(r) + f'(r)co_\lambda(r))\,|\partial_r^\top|^2 + f'(r)\lambda\frac{s_\lambda}{c_\lambda}(r)\,|(J\partial_r)^\top|^2\\
&\qquad - mf'(r)\,(co_\lambda(r) + \langle H, \partial_r\rangle).
\end{aligned} \tag{2.7}$$

3. Proof of Theorem 1.1

Case $\lambda > 0$

We take as f the solution $f = -(1/\lambda)c_\lambda$ of the equation

$$-f'' + f'\frac{c_\lambda}{s_\lambda} = 0$$

and taking account of $|(J\partial_r)^\top| \leq 1$ and $|\langle \partial_r, H\rangle| \leq |H|$, we get, from (2.7),

$$\Delta(-(1/\lambda)c_\lambda(r)) \leq \lambda\frac{s_\lambda(r)^2}{c_\lambda(r)} - mc_\lambda(r) + ms_\lambda(r)|H|. \tag{3.1}$$

Now, from the hypothesis on $|H|$ and the fact that the function $(m-1)(c_\lambda/s_\lambda) + (c_{4\lambda}/s_{4\lambda})$ is decreasing on the interval $(0, \rho)$, we have

$$\Delta(-(1/\lambda)c_\lambda(r)) \leq \lambda\frac{s_\lambda(r)^2}{c_\lambda(r)} - mc_\lambda(r) + s_\lambda(r)\left((m-1)\frac{c_\lambda(r)}{s_\lambda(r)} + \frac{c_{4\lambda}(r)}{s_{4\lambda}(r)}\right) = 0. \tag{3.2}$$

From Hopf principle, this implies that $\Delta(-(1/\lambda)c_\lambda)(r) = 0$, and hence the inequalities in (2.7), (3.1), and (3.2) must be equalities. Then, the function $r : M \longrightarrow \mathbb{R}$ must be the constant ρ, which amounts to say that $\psi(M) \subset \partial B_\rho(p)$, $|(J\partial_r)^\top| = 1$ (i.e. $J\partial_r$ is tangent to M), $\langle H, \partial_r\rangle = -|H|$ and $|H| = (m-1)co_\lambda(\rho) + co_{4\lambda}(\rho)$ (i.e. $H = \{-(m-1)co_\lambda(\rho) - co_{4\lambda}(\rho)\}\partial_r$).

If $m = 2n-1$, $\psi(M) = \partial B_\rho(p)$, and the equality in (2.7) implies $S(r) = S_\lambda(r)$ in $B_\rho(p)$, and from Lemma 2.1 we get that $B_\rho(p)$ is isometric to a geodesic ball of radius ρ in $\mathbb{K}^n(\lambda)$. From these facts it follows that ψ is a local isometry between M and $\partial B_\rho^{\lambda,n}$. Then (cfr. [3], page 150) ψ is a riemannian covering, but, since $\partial B_\rho^{\lambda,n}$ is simply connected, ψ must be an isometry. This finishes the proof of Theorem 1.1 when $\lambda > 0$.

Case $\lambda \leq 0$

Given any vector subbundle E of the pullback $\psi^*(T\overline{M})$ of $T\overline{M}$ by ψ, let us denote by π_E the projector on E. Since $J\mathcal{N}M \subset TM$, we have

$$|\partial_r^\top|^2 + |(J\partial_r)^\top|^2 = 1 - |\pi_{\mathcal{N}M}(\partial_r)|^2 + 1 - |\pi_{J\mathcal{N}M}(\partial_r)|^2 = 2 - |\pi_{\mathcal{N}M\oplus J\mathcal{N}M}\partial_r|^2. \tag{3.3}$$

Now, let us take as f the solution $f = -(1/(4\lambda))c_{4\lambda}$ (or $f = \frac{1}{2}t^2$ if $\lambda = 0$) of the equation

$$-f'' + f'\frac{c_\lambda}{s_\lambda} = f'\lambda\frac{s_\lambda}{c_\lambda}.$$

Then, using (3.3), the hypothesis $\lambda \leq 0$, and the other facts we used when $\lambda > 0$, we get

$$\begin{aligned}\Delta\left(-\frac{1}{4\lambda}c_{4\lambda}\right) &\leq \lambda s_\lambda^2(2 - |\pi_{\mathcal{N}M\oplus J\mathcal{N}M}\partial_r|^2) - mc_\lambda^2 - ms_\lambda c_\lambda\langle H, \partial_r\rangle \\ &\leq \lambda s_\lambda^2 - mc_\lambda^2 + ms_\lambda c_\lambda|H| \leq 0\end{aligned} \tag{3.4}$$

From (3.4), the theorem follows by an argument similar to that of the case $\lambda > 0$.

4. Proof of Corollaries 1.2 and 1.3

To begin with, let us recall the definition of the Hopf fibration $\Pi : \partial B^{\lambda,n}_\rho \longrightarrow \mathbb{C}P^{n-1}(1/(s^2_\lambda(\rho)))$. Let p be the centre of $\partial B^{\lambda,n}_\rho$, let $\exp_p$ be the exponential map $T_p\mathbb{K}^n(\lambda) \longrightarrow \mathbb{K}^n(\lambda)$, and let S^{2n-1} denote the unit sphere in $T_p\mathbb{K}^n(\lambda)$. From the knowledge of the Jacobi fields of $\mathbb{K}^n(\lambda)$ it follows that, for every $\xi \in S^{2n-1}$, $\exp_{p*\rho\xi} J\xi = s_{4\lambda}(\rho) J\partial_r(\exp_p(\rho\xi))$ and, for every $X, Y \in T_\xi S^{2n-1}$ orthogonal to $J\xi$, $\exp_{p*}X$ and $\exp_{p*}Y$ are orthogonal to $J\partial_r$ and satisfy $\langle \exp_{p*}X, \exp_{p*}Y\rangle = s^2_\lambda(\rho)\langle X, Y\rangle$ and $exp_{p*}JX = s_\lambda(\rho) J \exp_{p*}X$. From now on we shall call horizontal vectors those in S^{2n-1} orthogonal to $J\xi$ and also those in $\partial B^{\lambda,n}_\rho$ orthogonal to $J\partial_r$. If we scale the metric of S^{2n-1} by multiplying by $s^2_\lambda(\rho)$, we have that $\exp_p$ is an isometry on horizontal vectors which commutes with J. Furthermore, it is a dilation of factor $c^2_\lambda(\rho)$ on the vertical vectors, and it preserves the orthogonality between horizontal and vertical vectors. Then, if we consider the Hopf map $F : S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}(1/(s^2_\lambda(\rho))$, then $\Pi = F \circ \exp_p^{-1} : \partial B^{\lambda,n}_\rho \longrightarrow \mathbb{C}P^{n-1}(1/(s^2_\lambda(\rho))$ is still a riemannian submersion which, restricted to horizontal vectors, commutes with J, and it is also called the Hopf fibration. The fibres of this submersion are the integral curves of $J\partial_r$, which are closed geodesics of length $2\pi s_{4\lambda}(\rho)$.

Now, we are going to prove Corollary 1.2. Let $m = 2n-2$. A key observation is that the condition $J\partial_r \in TM$ implies that the integral curves of the vector field $J\partial_r$ which are in $\partial B^{\lambda,n}_\rho$ and have some point in $\psi(M)$ are completely contained in $\psi(M)$. More precisely: if c is an integral curve of $J\partial_r$ in M, then $\widetilde{c} = \psi \circ c$ is an integral curve of $J\partial_r$ in $\partial B^{\lambda,n}_\rho$, and hence it is a closed geodesic in $\partial B^{\lambda,n}_\rho$. If $p \in \widetilde{c}(\mathbb{R})$, then $\psi^{-1}(p)$ is finite, because ψ is an immersion and M is compact. Then c is closed (otherwise, $\psi^{-1}(\widetilde{c}(0))$ would be infinite because $\widetilde{c}$ is closed). All the integral curves of $J\partial_r$ in M being closed, and hence compact, they determine a regular foliation $\mathscr{F}$ on M which defines a quotient manifold $G = M/\mathscr{F}$ of dimension $2n - 3$.

Now we claim that we can define a metric on G so that the quotient map $\pi : M \longrightarrow G$ be a riemannian submersion. In fact, for every $q \in M$, we consider the orthogonal decompositions $T_qM = T_q\mathscr{F} \oplus H_q$ and $T_{\psi(q)}\partial B^{\lambda,n}_\rho = <J\partial_r> \oplus \mathscr{H}_{\psi(q)}$ defining the distributions H and $\mathscr{H}$ ($<J\partial_r>$ is the 1-dimensional distribution generated by $J\partial_r$). From Theorem 1.1, ψ is an isometric immersion from M into $\partial B^{\lambda,n}_\rho$, and this implies that $\psi_{*q}(H_q) \subset \mathscr{H}_{\psi(q)}$. Since Π is a riemannian submersion with fibres the integral curves of $J\partial_r$, if $\widetilde{\phi}_t$ denotes the flow of the vector field $J\partial_r$ in $\partial B^{\lambda,n}_\rho$, then $\widetilde{\phi}_{t*}$ takes the distribution $\mathscr{H}$ isometrically into itself. Moreover, if ϕ_t is the flow of $J\partial_r$ in M, then the above argument showing that $\widetilde{c} = \psi \circ c$ says that $\psi \circ \phi_t = \widetilde{\phi}_t \circ \psi$ and $\psi_* \circ \phi_{t*} = \widetilde{\phi}_{t*} \circ \psi_*$. From these facts it follows easily that ϕ_{t*} is an isometry from H_q onto $H_{\phi_t(q)}$. This shows that the metric we claimed can be defined. We will consider G endowed with this metric.

Since the images by ψ of the leaves of $\mathscr{F}$ are the fibres of the Hopf submersion Π, we can define the isometric immersion $\varphi : G \longrightarrow \mathbb{C}P^{n-1}(1/s^2_\lambda(\rho))$ by $\varphi(\pi(x)) = \Pi(\psi(x))$. This definition and the fact that the fibres of the submersion Π are the integral curves of $J\partial_r$ in $\partial B^{\lambda,n}_\rho$ implies that $\Pi^{-1}(\varphi(G)) = \psi(M)$.

In order to prove that φ is minimal, we will first show that the immersion $\psi : M \longrightarrow \partial B_\rho^{\lambda,n}$ is minimal.

Let us observe that, for $\lambda > 0$, the hypothesis $m = 2n - 2$ implies that the condition $J\mathcal{N}M \subset TM$ is also satisfied. In fact, we know that, on M, $\partial_r \in \mathcal{N}M$ and $J\partial_r \in TM$. Now, let $X \in \mathcal{N}M$ and assume it is orthogonal to ∂_r. We have $\langle JX, \partial_r \rangle = -\langle X, J\partial_r \rangle = 0$ and, obviously, $\langle X, JX \rangle = 0$, then $JX \in TM$, and the claim is proved.

Now, let $\{e_{2n-1} = \partial_r, e_{2n}\}$ be a local orthonormal frame of $\mathcal{N}M$. The remark above allows us to take a local orthonormal frame of TM of the form $\{e_1 = J\partial_r, e_2 = Je_{2n}, e_3, \ldots, e_{2n-2}\}$.

If α is the second fundamental form of the immersion $\psi : M \longrightarrow \mathbb{K}^n(\lambda)$, then, denoting by $^\perp$ the component orthogonal to M of any vector, we have

$$\alpha(J\partial_r, J\partial_r) = (J\overline{\nabla}_{J\partial_r}\partial_r)^\perp = (J(co_{4\lambda}(\rho)J\partial_r))^\perp = -co_{4\lambda}(\rho)\partial_r. \tag{4.1}$$

Since $\psi(M) \subset \partial B_\rho^{\lambda,n}(p)$, we can consider the second fundamental forms $\widetilde{\alpha}$ of the immersion of M in $\partial B_\rho^{\lambda,n}(p)$ and $\widetilde{\widetilde{\alpha}}$ of the immersion of $\partial B_\rho^{\lambda,n}(p)$ in $\mathbb{K}^n(\lambda)$. Then

$$\begin{aligned}\sum_{i=1}^{2n-2} \widetilde{\alpha}(e_i, e_i) &= \sum_{i=1}^{2n-2} \{\alpha(e_i, e_i) - \widetilde{\widetilde{\alpha}}(e_i, e_i)\} \\ &= (2n-2)H - \{-co_{4\lambda}(\rho) - (2n-3)co_\lambda(\rho)\}\partial_r = 0,\end{aligned} \tag{4.2}$$

i.e., the immersion of M in $\partial B_\rho^{\lambda,n}$ is minimal, but this implies that $\varphi : G \longrightarrow \mathbb{C}P^{n-1}(1/s_\lambda^2(\rho))$ is minimal (cfr. [13], Lemma 2). This finishes the proof of Corollary 1.2.

Corollary 1.3 follows from 1.2. In fact, if $n = 2$, G has dimension 1 and is compact, then, the minimal immersion φ is the parametrization of a closed geodesic of $\mathbb{C}P^1(1/s_\lambda^2(\rho))$, which is a great circle, that is, a geodesic sphere of radius $(\pi/4)s_\lambda(\rho)$ around some point $q \in \mathbb{C}P^1(1/s_\lambda^2(\rho))$, then $\psi(M) = \Pi^{-1}(\varphi(G))$ is a tubular hypersurface of $\partial B_\rho^{\lambda,n}$ of radius $(\pi/4)s_\lambda(\rho)$ around $\Pi^{-1}(q)$, which is an integral curve of $J\partial_r$. From the facts that $\varphi(G)$ is a geodesic, $\psi(M) = \Pi^{-1}(\varphi(G))$ and Π is a riemannian submersion with totally geodesic fibres, it follows that M has two orthogonal totally geodesic foliations, so it is locally the product of two curves and, therefore, it is flat and since it admits two globally well defined linearly independent vector fields, it is a torus. To end the proof we observe that $\psi : M \longrightarrow \psi(M)$ is a local isometry, then (cfr. [3], page 150) it is a riemannian covering, and M must be also a flat torus.

5. Further results

Theorem 1.1 is, in some sense, complete for M of codimension 1 or M of any codimension and $\lambda > 0$. However, for $\lambda < 0$ and M of codimension greater than

1, the additional condition $J\mathcal{N}M \subset TM$ lead us to think that there could be immersions $\psi : M \longrightarrow \mathbb{C}H^n(\lambda)$ which do not satisfy the above condition, but having its image contained in a geodesic ball of radius ρ and the maximum of the norm of its mean curvature strictly less than $(1/m)\{(m-1)co_\lambda(\rho)+co_{4\lambda}(\rho)\}$. The next proposition shows that this maximum of $|H|$ must be strictly greater than $co_\lambda(\rho)$ for "small" codimension and greater than or equal to $co_\lambda(\rho)$ for "big" codimension.

Theorem 5.1. *Let M be a compact Riemannian manifold of dimension m. Let $\psi : M \longrightarrow \mathbb{C}H^n(\lambda)$, $\lambda < 0$, be an isometric immersion such that $\psi(M) \subset B_\rho^{\lambda,n}$ and $|H| \le co_\lambda(\rho)$. Then $1 \le m \le n-1$, $|H| = co_\lambda(\rho)$ and there exists a minimal totally real isometric immersion $\varphi : M \longrightarrow \mathbb{C}P^{n-1}(1/s_\lambda^2(\rho))$ such that ψ is a lifting of φ by Π.*

Proof . Taking $f = -(1/\lambda)c_\lambda$ in (2.7), we get

$$\Delta\left(-\frac{1}{\lambda}c_\lambda\right) \le \lambda\frac{s_\lambda^2}{c_\lambda}|(J\partial_r)^\top|^2 - mc_\lambda - m\langle H,\partial_r\rangle s_\lambda \le -m\,c_\lambda + m\,|H|s_\lambda \le 0.$$

Again by Hopf principle, the above inequalities must be equalities, and hence

(a) ∂_r, $J\partial_r \in \mathcal{N}M$, and

(b) $H = -co_\lambda(\rho)\partial_r$.

These conditions imply that

(c) the immersion ψ is minimal in $\partial B_\rho^{\lambda,n}$.

$\psi_*(TM)$ must be orthogonal to ∂_r and $J\partial_r$. Let O be the tensor of the riemannian submersion Π defined on horizontal vectors by $O_X Y = (1/2)v[X,Y]$, v being the projection onto the vertical distribution (in this case, the projection on $<J\partial_r>$). It is well known that, for horizontal basic vector fields X, Y, one has $O_X Y = v\widetilde{\nabla}_X Y$, where $\widetilde{\nabla}$ is the riemannian connection on $\partial B_\rho^{\lambda,n}$ (see [15] or [7]). Then, denoting by $\widetilde{\alpha}$ the second fundamental form of $\partial B_\rho^{\lambda,n}$ in $\mathbb{C}H^n(\lambda)$, and taking as X, Y the restriction to $\psi_*(M)$ of local horizontal basic vector fields, we get

$$\begin{aligned}\frac{1}{2}v[X,Y] = O_X Y = v\widetilde{\nabla}_X Y &= \langle\overline{\nabla}_X Y, J\partial_r\rangle J\partial_r = -\langle\overline{\nabla}_X(JY),\partial_r\rangle J\partial_r\\ &= \langle\widetilde{\alpha}(X,JY),\partial_r\rangle J\partial_r = -co_\lambda(\rho)\langle X,JY\rangle J\partial_r.\end{aligned}$$

Then $v[X,Y] = 0$ if and only if $\langle X,JY\rangle = 0$. Since the Lie bracket of two local vector fields tangent to $\psi_*(TM)$ must be tangent to $\psi_*(TM)$, we find that, for all X,Y tangent to $\psi_*(M)$, $\langle X,JY\rangle = 0$, i.e., $JTM \subset \mathcal{N}M$, which implies $1 \le m \le n-1$. Finally, we only need to observe that the condition $JTM \subset \mathcal{N}M$ is equivalent to say that $\Pi_*(\psi_{*q}(M))$ is totally real in $T_{\Pi(\psi(q))}\mathbb{C}P^{n-1}$ for every $q \in M$ and condition (c) is equivalent to say that $\Pi(\psi(M))$ is minimal in $\mathbb{C}P^{n-1}$ ([13]).

Remark . For $n = 2$, the above theorem says that the only isometric immersions in $\mathbb{C}H^2(\lambda)$ contained in $B_\rho^{\lambda,2}$ with $|H| \le co_\lambda(\rho)$ are horizontal geodesics of $\partial B_\rho^{\lambda,2}$.

I. Castro and F. Urbano [2] have classified all the totally real minimal tori in $\mathbb{C}P^2$ which are invariant under a 1-parameter group of holomorphic isometries. This provides examples of immersions satisfying the hypotheses of Theorem 5.1, for n=3 and m=2.

Let us observe that

$$m\, co_\lambda(\rho) < (m-1)\, co_\lambda(\rho) + co_{4\lambda}(\rho) \text{ if } \lambda < 0$$

and

$$m\, co_\lambda(\rho) > (m-1)\, co_\lambda(\rho) + co_{4\lambda}(\rho) \text{ if } \lambda > 0$$

For the case $\lambda < 0$ we have got a Theorem (5.1) with the smaller bound $m\, co_\lambda(\rho)$ and another Theorem (1.1) with the greater bound, the latter with the additional restriction that $J\mathcal{N}M \subset TM$. This condition has the effect of making $J\partial_r$ tangent to the image of M.

For $\lambda > 0$, the bound given in Theorem 1.1 is the smaller one. Then it is natural to ask if there is another theorem with the greater bound $m\, co_\lambda(\rho)$, with some additional hypothesis. As in the case of negative λ, this condition must force the image of M to be tangent to the directions of maximal normal curvature in the geodesic sphere of $\mathbb{K}^n(\lambda)$. For $\lambda > 0$ the direction having minimal normal curvature is $J\partial_r$, which then should be avoided. Thus, the natural condition would be $J\mathcal{N}M \subset \mathcal{N}M$, i.e. that ψ be a complex immersion. But it is known (cfr. [8]) that a compact Kähler submanifold immersed in $\mathbb{C}P^n(\lambda)$ intersects every totally geodesic complex hypersurface $\mathbb{C}P^{n-1}(\lambda)$ in $\mathbb{C}P^n(\lambda)$, and therefore it cannot be contained in any proper geodesic ball.

References

1. Choi, H.I., Schoen, R.: The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature. Invent. Math. **81** (1985) 387–394
2. Castro, I., Urbano, F.:New examples of minimal Lagrangian tori in the complex projective plane. Manuscr. Math. **85** (1994) 265–281
3. Carmo, M.P. do: Riemannian Geometry. Birkhäuser, Boston 1992
4. Eschenburg, J.H., Heintze, E.: Comparison theory for Riccati equations. Manuscr. Math. **68** (1990) 209–215
5. Giménez, F.: Comparison theorems for the volume of a complex submanifold of a Kaehler manifold. Isr. J. Math. **71** (1990) 239–255
6. Gray, A.: Tubes. Addison-Wesley, New York Reading 1990
7. Gray, A.: Pseudo-Riemannian almost product manifolds and submersions. Journal of Mathematics and Mechanics **16** (1967) 715–738
8. Goldberg, S.I., Kobayashi, S.: Holomorphic bisectional curvature. J. Differ. Geom. **1** (1967) 225–233
9. Greene, R.E., Wu, H.: Function Theory on Manifolds Which Possess a Pole. (Lecture Notes in Mathematics, vol. 699) Springer-Verlag, Berlin, 1979
10. Jorge, L. P., Koutroufiotis, D.: An estimate for the curvature of bounded submanifolds. Am. J. Math. **103** (1981) 711–725
11. Jorge, L. P., Xavier, F.V.: An inequality between the exterior diameter and the mean curvature of bounded immersions. Math. Z. **178** (1981) 77–82

12. Kowalski, O., Vanhecke, L.: A new formula for the shape operator of a geodesic sphere and its applications. Math. Z. **192** (1986) 613–625
13. Lawson, Jr., H. B.: Rigidity Theorems in rank-1 symmetric spaces. J. Differ. Geom. **4** (1970) 349–357
14. Markvorsen, S.: A Sufficient Condition for a Compact Immersion to be Spherical. Math. Z. **183** (1983) 407–411
15. O'Neill, B.: The fundamental equations of a submersion. Mich. Math. J. **13** (1966) 459–469
16. Royden, H.L.: Comparison theorems for the matrix Riccati equation. Commun. Pure Appl. Math. **XLI** (1988) 739–746

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